Generalized Derivations in Prime Rings with Involution

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Abstract
The purpose of this paper is to find the commutativity of prime rings with involution of second kind constrained with generalized derivation. In fact if R is a 2-torsion free prime ring admits generalized derivation $F: R \to R$ associated with derivation $d: R \to R$ then ring behaviour is examined when one of the following identities holds: (1) $[F(x), F(x^*)] - x \circ x^*=0$; (2) $[F(x), d(x^*)] - x \circ x^*=0$.

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I. INTRODUCTION

Throughout this paper, R will represent a ring with multiplicative centre Z(R). For any pair of elements $x, y \in R$, the commutator (resp. anti commutator) is defined by $[x, y]$ (resp. $x \circ y$) which is equal to $xy - yx$ (resp. $xy + yx$. A ring R is 2-torsion free if for any $x \in R$, $2x = 0$ implies $x = 0$. A ring R is said to be a prime ring if $aRb = 0 \Rightarrow a = 0$ or $b = 0$. An additive mapping $d: R \to R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive map $F: R \to R$ is called generalized derivation constrained with derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $*: R \to R$ is an involution if $(x^*)^* = x$ and $xy^* = y^*x^*$. An element $x \in R$ is symmetric (resp. skew symmetric) if $x^* = x$ (resp. $x^* = -x$). The sets of all symmetric elements and skew symmetric elements in a ring R are H(R) and S(R) respectively. An involution is said to be of first kind if $Z(R) \subseteq H(R)$ and it is said to be of second kind if $Z(R) \cap S(R) \neq 0$. Motivated by Nadeem et al. [6] the possibilities in the behaviour of prime ring with involution of second kind constrained with generalized derivation are examined.

II. Preliminary Results

Lemma 2.1 ([6], Lemma 2.2) Let R be a 2-torsion free prime ring with involution of second kind and if R admits a derivation $d: R \to R$ such that $[d(h), h] = 0$ for all $h \in H(R)$, then $d(Z(R)) = (0)$.

Lemma 2.2 Let R be a 2-torsion free ring with involution of second kind such that $x \circ x^*=0$ for all $x \in R$. Then R is commutative.

Proof. We have

$$x \circ x^*=0 \text{ for all } x \in R. \quad (1)$$

By linearizing (1), we have

$$x \circ y^* + y \circ x^*=0 \text{ for all } x, y \in R. \quad (2)$$

Replacing y by y^*, we get

$$x \circ y + y^* \circ x^*=0 \text{ for all } x, y \in R. \quad (3)$$

Replacing *y* by *x* in (3), we get

$$x^2 + (x^*)^2 = 0 \text{ for all } x \in R. \tag{4}$$

Taking $y \in Z(R)\backslash\{0\}$ and x by x^2, in (3), we get

$$x^2 \circ y + y^* \circ (x^*)^2 = 0 \text{ for all } x, y \in R. \tag{5}$$

Using R is a 2-torsion free implies

$$x^2 y + (x^*)^2 y^* = 0 \text{ for all } x, y \in R. \tag{6}$$

Using (4) and (6) together, we get

$$x^2(y - y^*) = 0 \text{ for all } x, y \in R. \tag{7}$$

As $(y - y^*) \in Z(R)$ implies either $y - y^* = 0$ for all $y \in Z(R)$ or $x^2 = 0$ for all $x \in R$.

If $y - y^* = 0$ then $y = y^*$ for all $y \in Z(R)$ which leads to contradict the involution of second kind and $x^2 = 0$ for all $x \in R$ implies R is commutative.

III. Main Results

Theorem 3.1 Let R be a prime ring with unity and R has involution of second kind with 2-torsion free. If R admits a generalized derivation $F: R \to R$ associated with derivation $d: R \to R$ such that $[F(x), F(x^*)] - x \,\square\, x^* = 0$ for all $x \in R$ then one of the following holds:

1. A prime ring R is commutative.

2. A prime ring R is non commutative subring of division ring Δ, and there exists $\delta \in \Delta$ such that $F(x) = \delta x + x\delta$ for all $x \in R$.

3. A prime ring R is non commutative subring of a 2×2 total matrix ring over a field, there exist $m \in M$, such that $F(x) = mx + xm$ for all $x \in R$.

Proof. By the given hypothesis, we have

$$[F(x), F(x^*)] - x \,\square\, x^* = 0 \text{ for all } x \in R. \tag{8}$$

Replacing x by $\square + k$ in equation (8) where $h \in H(R)$ and $k \in S(R)$, we have:

$$[F(\square), F(\square)] + [F(k), F(\square)] + [F(\square), F(-k)] + [F(k), F(-k)]$$
$$-\square \,\square\, \square - k \,\square\, \square - \square \,\square\, (-k) - k \,\square\, (-k) = 0.$$

Using (8), we get

$$2[F(k), F(h)] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

Since $\mathrm{Char}(R) \neq 2$, implies

$$[F(k), F(h)] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{9}$$

Replacing h by k_c^2 in (9), where $k_c \in S(R) \cap H(R)$, we get

$$[F(k), F(k_c)]k_c = 0 \text{ for all } k \in S(R) \text{ and } k_c \in S(R) \cap Z(R).$$

Since $k_c \in Z(R)$ implies

$$[F(k), F(k_c)] = 0 \text{ for all } k \in S(R) \text{ and } k_c \in S(R) \cap Z(R). \tag{10}$$

Replacing k by $h_0 k_c$ in (10) where $h_0 \in H(R)$ and $k_c \in S(R) \cap Z(R)$, we get

$$[F(h_0 k_c), F(k_c)] = 0 \text{ for all } h_0 \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

which further implies

$$[F(h_0)k_c + h_0 d(k_c), F(k_c)] = 0 \text{ for all } h_0 \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

Equation (9) and primeness gives

$$[\square_0, F(k_c)]d(k_c) = 0 \text{ for all } \square_0 \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

Using $d(k_c) \in Z(R)$, we get

$$[h_0, F(k_c)] = 0 \text{ or } d(k_c) = 0 \text{ for all } h_0 \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

First assume that $d(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$. Replacing k_c by $\square_0 k_c$ in (9) where $\square_0 \in H(R)$ and $k_c \in S(R) \cap Z(R)$ and using $d(k_c) = 0$, we get

$$[F(\square_o), F(\square)]k_c = 0 \text{ for all } \square_0, \square \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

Again using primeness of R, we get

$$[F(\square_o), F(\square)] = 0 \text{ for all } \square_0, \square \in H(R)$$

(11)

Using equation (9) and (11), we get

$$[F(x), F(\square)] = 0 \text{ for all } \square_0 \in H(R) \text{ and } x \in R \tag{12}$$

Again replacing $\square$ by kk_c in (12) and proceeding as above, we get

$$[F(x), F(y)] = 0 \text{ for all } x, y \in R.$$

In the view of ([3], Theorem 3.4) we get desired result in this case.

Now if

$$[k_o, F(k_c)] = 0 \text{ for all } \square_0 \in H(R) \text{ and } k_c \in S(R) \cap Z(R). \tag{13}$$

Replacing h_0 by $k_0 k_c$ where $k_0 \in S(R)$ and $k_c \in S(R) \cap Z(R)$ and using primeness, we get

$$[k_o, F(k_c)] = 0 \text{ for all } k_o \in S(R) \text{ and } k_c \in S(R) \cap Z(R). \tag{14}$$

Since every element of x can be uniquely represent as $2y = \square + k$ where $\square \in H(R)$ and $k \in S(R)$ and then using (13) and (14), we get

$$2[y, F(k_c)] = 0 \text{ for all } y \in R \text{ and } k_c \in S(R) \cap Z(R),$$

implies $F(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$. Replacing k by $\square k_c$ in (9) $\square \in H(R)$ and $k_c \in S(R) \cap Z(R)$, we get

$$[\square, F(\square)]d(k_c) = 0 \text{ for all } h \in H(R) \text{ and } k_c \in S(R) \cap Z(R).$$

Again using primeness of R, we ge either $[h, F(h)] = 0$ for all $h \in H(R)$ or $d(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$. The case $d(k_c) = 0$ is discussed above.

So we left with the case get $[h, F(h)] = 0$ for all $h \in H(R)$. Replacing h by $k_c k$ in (9) where $k \in S(R)$ and $k_c \in S(R) \cap Z(R)$. We get $[F(k_c k), F(k)] = 0$ where $k \in S(R)$ and $k_c \in S(R) \cap Z(R)$. Using $F(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$ and $S(R) \cap Z(R) \neq 0$ implies $[k, F(k)] = 0$ for all $k \in S(R)$. Replace k by $k_c h$ we get $[h, d(h)]k_c^2 = 0$. Again using primeness and $S(R) \cap Z(R) \neq 0$, implies $[h, d(h)] = 0$ for all $h \in H(R)$. In view of Lemma 2.1 we have at $d(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$ applying same process we get our result.

Theorem 3.2. Let R be a prime ring has involution of second kind with 2- torsion free. If R admits a generalized derivation $F : R \to R$ associated with derivation $d : R \to R$ such that $[F(x), d(x^*)] - x \circ x^* = 0$ for all $x \in R$ then R is commutative.

PType equation here.roof. If $d = 0$, then we have $x \circ x^* = 0$. By Lemma 2.2 we get R is commutative.

By the given hypothesis, we have

$$[F(x), d(x^*)] - x \circ x^* = 0 \text{ for all } x \in R. \tag{15}$$

Replacing x by $x + y$ in equation (15), for all $x, y \in R$, we have

$$[F(x), d(y^*)] + [F(y), d(x^*)] = 0 \quad \text{for all } x, y \in R. \tag{16}$$

Replacing x by k_c and y by x^* in (16), we get

$$[F(k_c), d(x)] = 0 \text{ for all } x \in R \text{ and } k_c \in S(R) \cap Z(R).$$

Thus by ([10], Theorem 2) we conclude that $F(k_c) \in Z(R)$ for all $k_c \in S(R) \cap Z(R)$.

Replacing x by $\square + k$ in (16), we get

$$[F(h), d(-k)] + [F(k), d(h)] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{17}$$

Replacing k by hk_c in (17) where $\square \in H(R)$ and $k_c \in S(R) \cap Z(R)$ in (17), then we have $[F(h), h]d(k_c) + [d(h), h]d(k_c) = 0$ for all $h \in H(R)$ and $k_c \in S(R) \cap Z(R)$.

Using primeness, we have either $d(k_c) = 0$ or $[F(h), h] + [d(h), (h)] = 0$ for all $h \in H(R)$ and $k_c \in S(R) \cap Z(R)$. Noting that $F(k_c) \in Z(R)$ and $F(k_c)x + (k_c)d(x) = F(k_c x) = F(xk_c) = F(x)k_c + xd(k_c)$ and $[F(k_c x), x] = [F(xk_c), x]$ for all $k_c \in S(R) \cap Z(R)$ and $x \in R$, we get $[F(x), x]0 = [d(x), x]$ for all $x \in R$. Using this $[F(h), h] + [d(h), (h)] = 0$ for all $\square \in H(R)$ becomes $[d(\square), \square] = 0$ for all $\square \in H(R)$ by Lemma 2.1 implies $d(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$.

Now if $d(k_c) = 0$ for all $k_c \in S(R) \cap Z(R)$. Replacing x by xk_c in (16) where $k_c \in S(R) \cap Z(R)$ and comparing with (15) we get $[F(x), d(y^*)] = 0$ for all $x, y \in R$. That is $[F(x), d(y)] = 0$ for all $x, y \in R$. Then ([10], Theorem 2) gives us $F(x) \in Z(R)$ for all $x \in R$ i.e. $F(R) \subset Z(R)$ implies $[F(x), y] = 0$ for all $x, y \in R$ implies $[F(x), x] = 0$ for all $x \in R$ using ([3], Lemma 2.2) we get R is commutative.

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