A Comparison of axisymmetric vibration of tapered annular circular plate with different linearly and parabolically varying thickness with different boundary conditions

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Abstract:

In this study, a vibration analysis of a tapered annular circular plate having different linearly and parabolically varying thickness with different taper ratios for free- free and clamped - free boundary conditions is investigated by keeping the total mass of the plate constant. FEM is used to determine the Eigen value of the plate. The present work is validated with the existing literature and good agreement of results is observed. Further, the mode variation and stiffness variation due to taper ratio is observed for all the thickness variations. Finally the solution for flexural response for actuation and suppression associated with different thickness variation is suggested.

I. INTRODUCTION

In engineering applications, annular circular tapered plates with different arbitrarily varying thickness are widely used in structural components i.e. diaphragms and deck plates in launch vehicles, diaphragms of turbines, telephone industry, aircrafts/missiles, naval structures, nuclear reactors, optical systems, constructions of ships, automobiles and other vehicles, space shuttle, sound emitters and receivers, circular ports and panels, pneumatic pumps, cochleae, gongs and cymbals etc. By appropriate tapering of the plate thickness with different combinations of arbitrarily varying thickness, these tapered plates can have significantly greater resistance to bending, buckling and vibration over the corresponding plates of uniform thickness. The plates with varying thickness with different combinations of arbitrarily varying thickness are used to alter the dynamic characteristic of structure due to change in stiffness. Hence, the vibration characteristic of such tapered plates has importance in practical design. Several researchers have examined the vibration response of circular or annular plate with tapered or uniform thickness.

II. LITERATURE SURVEY

Wang et al. [1] determined the elastic buckling of tapered plate using shooting method and Rayleigh - Ritz approach. Gupta and Goyal [2] determined the forced asymmetric response of linearly tapered circular plate using classical plate theory. Laura et al. [3] used the optimized Rayleigh – Ritz method to determine the buckling of circular annular plates having non uniform thickness. Sharma et al. [4] used the Ritz method to determine the transverse vibration of free annular circular plate with linearly varying thickness. Wang [5] determined the vibration modes of concentrically supported free circular plate using asymptotic formula. Singh and Hassan [6] used the Rayleigh – Ritz method to determine the transverse vibration of a circular plate with arbitrary thickness variation. Duan et al. [7] determined the vibration modes of circular plates with free edge using Mindlin plate theory. Gupta et al. [8] determined the Vibration of non-homogeneous circular Mindlin plates with variable thickness using chebyshev collocation technique. Kang [9] determined the Three-dimensional vibration analysis of thick, circular and annular plates with nonlinear thickness variation using Ritz method. Leissa (10) determined mode and natural frequencies of plate at various geometry condition and boundary condition. Mindlin et al. (11) used sixth order thick plate theory to describe flexural vibration of thick circular disk. Mcgee et al. (12) used the modified Bessel function to solve free vibration of thick annular sector plate with simply supported radial edges. Lee et al. (13) determines the natural frequencies from the out of plane modes of annular circular plate by thick and thin plate theories. Irie et al. (14) studied out of plane vibration of annular Mindlin plate with varying thickness by transfer matrix approach. Prasad et al. (15) investigated the axisymmetric vibrations of circular plates of linearly varying thickness using Frobenius method. Luisoni et al. (16) investigated the antisymmetric modes of vibration of a circular plate elastically restrained against rotation and of linearly varying thickness using classical plate theory. Grossi and Laura (17) investigated the transverse vibrations of circular plates of linearly varying thickness using Galerkin's method. Singh and Sexana (18) determined the axisymmetric vibration of a circular plate with double linear variable thickness using Rayleigh – Ritz method. Laura and De Greco (19) have written a note on vibrations and elastic stability of circular plates with thickness varying in bilinear fashion using Rayleigh – Ritz method. Singh and Chakraverty (20) determined the transverse vibration of circular and elliptic plates with quadratically varying thickness using Rayleigh – Ritz method. Barakat and Baumann (21) investigated the axisymmetric vibrations of a thin circular plate having parabolic thickness variation using Ritz Galerkin method. Lenox and Conway (22) found an exact closed form solution for the flexural vibration of a thin annular plate having a parabolic thickness variation using Bessel function. Avalos et al. (23) investigated the transverse vibrations of a circular plate carrying an elastically mounted mass using Rayleigh – Ritz method. Gupta et al. (24) found the buckling and vibration of polar orthotropic annular plates of variable thickness using Spline technique. Chen (25) found the axisymmetric vibration of circular and annular plates with arbitrarily varying thickness using Finite Element Method. Gorman (26) investigated the natural frequencies of transverse vibration of polar orthotropic variable thickness annular plates using Finite Element Method.

Review of the literature suggested that a comparative study of axisymmetric vibration of tapered annular circular plate with linearly and parabolically varying thickness by keeping the mass of the plate constant with free- free and clamped - free boundary conditions is not reported. Therefore, in this paper an attempt has been made to find out the vibration response of an annular circular plate with different combinations of linearly and parabolically varying thickness for free- free and clamped - free boundary conditions with different taper ratio by keeping the mass constant.

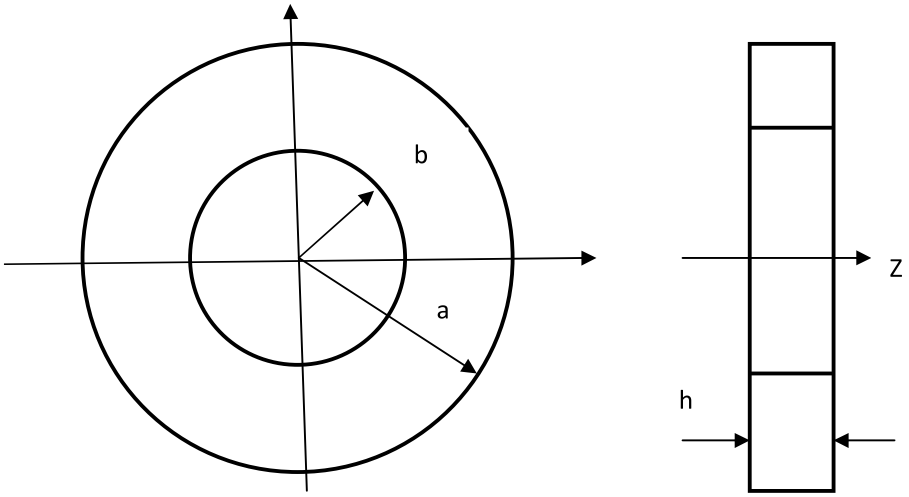


Fig.1. Geometry and co-ordinate system of annular circular plate

III. PROPOSED METHOD

2. Mathematical formulation

2.1. Plate free vibration

Natural frequency and modes shape of the plate to solve the eigen value problem for ω^2 is given by Eq. 1:

$$\left([k]-\omega^2[M]\right)\psi_{mn}=0 \tag{1}$$

where [k] is the stiffness matrix and [M] is the mass matrix while ψ_{mn} is the mode shape of structure and ω is the corresponding natural frequency of the plate in rad/sec.
The non-dimensional frequency parameter 'λ^2'is given by Eq.2:

$$\lambda^2 = \omega a^2 \sqrt{\frac{\rho\, h}{D}} \tag{2}$$

where D, the flexure rigidity = $\frac{Eh^3}{12(1-\upsilon^2)}$, a = outer radius, E = Young's modulus of elasticity, υ = Poisson's ratio, h = thickness of the plate and ρ = density of plate.

2.2. Thickness variation of the plate

A thick annular plate with different linear and parabolic varying thickness is considered for analysis and is reported in Fig. 2. The thickness of the plate is varied in radial direction by keeping its total mass unchanged. The plate thickness at radial direction is given by hx = h [1 – T_x {f(x)} ^{n}], where 'h' is considered to be maximum plate thickness and f(x) is considered to be an arbitrary function of ordinate x.
where,

$$f(x) = \begin{cases} 0, & x = b \\ 1, & x = a \end{cases} \quad \text{and} \quad f(x) = \frac{x-b}{a-b} \quad \text{where } b<x< a \tag{3}$$

and the taper parameter or taper ratio (T_x) is given by Eq .4:

$$T_x = (1 - h_{min}/h) \tag{4}$$

The equations for Case I and Case II (Fig.2) with decreasing thickness variation is given by Eq. 5:

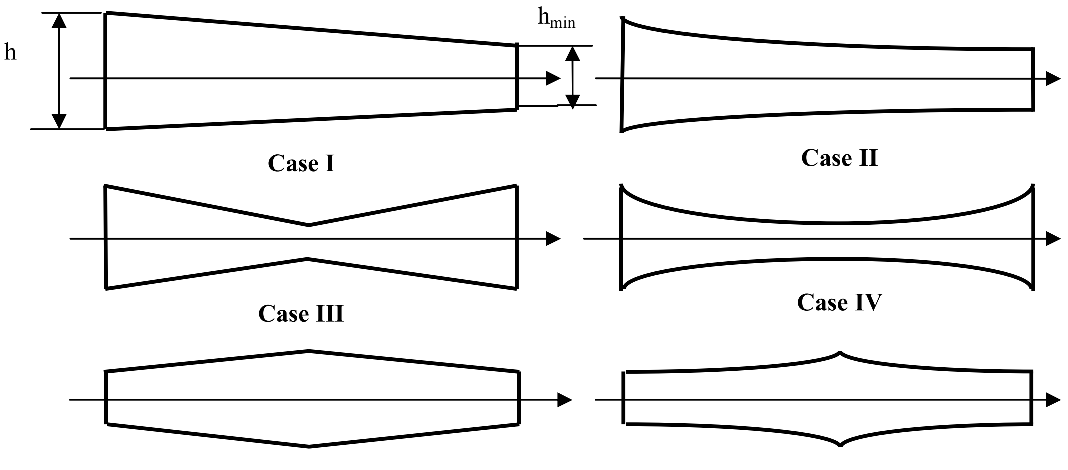
$$h_x = h\ \{1 - T_x(x - b/a - b)^2\} \tag{5}$$

The equations for Case III and Case IV (Fig.2) with (decreasing-increasing) and Case V and Case VI (Fig.2) with (increasing -decreasing) thickness variation are given by Eqs. (6 - 7):

$$h_x = h\{1 - T_x\ (1 - \text{abs}\ (1 - 2(x\text{-}b)/a\text{-}b))\ ^2\} \tag{6}$$
$$h_x = h\{1 - T_{x.}\text{abs}\ (1 - 2(x\text{-}b)/(a\text{-}b)^2\} \tag{7}$$

where, n = 1 for linear profile and n= 2 for parabolic profile. The total volume of the plate is kept constant and is given by Eq. 8:

$$\text{Volume} = \pi(a^2 - b^2)h = \int_b^a (a^2 - b^2)\ h_x dx \tag{8}$$



Case I

Case II

Case III

Case IV

Case V

Case VI

Fig. 2. Plate with different linearly and parabolically varying thickness variations.

In this paper, effect of non dimensional frequency parameter on flexural response due to out of plane modes for an annular circular plate with different combinations of arbitrarily varying thickness with different linear and parabolic varying thickness with different taper ratios of 0.25, 0.50 and 0.75 for clamped - free and free – free boundary conditions keeping the total mass of the plate constant are analyzed Six arrangements of plate with different combination of arbitrarily varying thickness are considered as shown in Fig.2. The selection of different combinations of arbitrarily varying thickness are such that the mass of uniform thickness plate is equal to mass of plate with arbitrarily varying thickness and in all six cases the total mass of the plate with arbitrarily varying thickness remains constant. The specification and the material properties of an annular circular plate are reported in Table 1. ANSYS has been used for numerical computation.

Table 1. The specification and the material properties of the annular circular plate

Dimension of the plate	Isotropic annular circular plate
Outer radius (a) m	0.1515
Inner radius (b) m	0.0825
Radii ratio, (b/a)	0.54
Thickness ratio, (h/a),	0.21
Density, ρ (kg/m^3)	7905.9
Young’s modulus, E (GPa)	218
Poisson’s ratio, υ	0.305

Table 2. Comparison and validation of frequency parameter λ^2 of uniform isotropic annular circular plate with free - free boundary conditions for taper ratio, $T_x = 0.00$ obtained in present work with that of Lee et al. [13].

Plate	Mode	Non dimensional frequency parameter, λ^2	
		H. Lee et al.[13]	Present work
Parabolic plate			
	(0,0)	3.82	3.83
b/a = 0.54	(0,1)	8.85	8.82
h/a = 0.21	(0,2)	10.59	10.02
	(0,3)	15.42	13.70

Table 3. Comparison and validation of frequency parameter λ^2 of uniform isotropic annular circular plate with clamped - free boundary conditions for taper ratio, $T_x = 0.00$ obtained in present work with that of Lee et al. [13].

Plate	Mode	Non dimensional frequency parameter, λ^2	
		H. Lee et al.[13]	Present work
Parabolic plate			
	(0,0)	13.61	13.49
b/a = 0.54	(0,1)	13.43	13.50
h/a = 0.21	(0,2)	15.28	14.12
	(0,3)	16.81	16.67

IV. RESULTS AND DISCUSSION

3.1. Validation of modal frequency

For validation of modal frequency of thick annular isotropic plate, the published result of Lee et al. [13] is taken as reported in Table 2 and 3. From Table 2 &3 it is clear that the results obtained in this paper matches well with the published results [13].

2.3 Flexural response of plate for different combinations of linearly and parabolically varying thickness with different taper ratio

The effects on natural frequency parameter (λ2) due to different combination of arbitrarily varying thickness for different taper ratio are investigated by keeping the mass of the plate the constant. Table 4 and Table 5 compares the first four natural frequency parameter λ^2 of Case I and Case II plate with linearly and parabolically varying thickness with different taper ratio for different boundary conditions using different methods. It is investigated from the Table 4 and Table 5 that the result obtained from analytical and numerical matches well with each other. It is further investigated from Table 4 and Table 5 that the non dimensional frequency parameters of Case I plate with linearly decreasing thickness variation reduce more in comparison to Case II plate with parabolically decreasing thickness variation for both free – free and clamped - free boundary conditions with different taper ratios. Fig.3 and Fig.4 compares the variation of natural frequency parameters of Case I and Case II plate with different modes with different radii ratios for free – free boundary conditions. It is clear from Fig.3 and Fig.4 that due to more stiffness, (0, 0), (0, 2) and (0, 3) modes tends to decreases with radii ratio while (0, 1) modes increases with increasing radii ratio due to less stiffness associate with the modes. However, similar pattern of curve is obtained for Case I and Case II plate with free - free boundary conditions. Similarly, from Fig.5 and Fig.6 it is observed that due to less stiffness associated with these modes, the modes (0, 0), (0, 1), (0, 2) and (0, 3) increases with increasing radii ratio for Case I and Case II plate with clamped – free boundary conditions and again a similar pattern of curve is obtained for Case I and Case II plate. Hence, it can be justified that the mode variation has a significant effect on vibration response. Table 6 and Table 7 compares the numerical comparison of the first four natural frequency parameter of the Case III, Case IV, Case V and Case VI plate with different linearly and parabolically varying thickness with different taper ratio for different

Table 4. Comparison of first four frequency parameter λ^2 of isotropic annular circular plate of linear varying thickness for different boundary conditions with different taper ratio using different methods.

BCS	Case	Mode	Non dimensional frequency parameter, λ^2			
			$T_x = 0.00$	$T_x = 0.25$	$T_x = 0.50$	$T_x = 0.75$
			NUM	NUM	NUM	NUM
Free- Free	I	(0,0)				
		(0,1)	3.83	3.64	3.44	3.24
		(0,2)	8.82	8.41	7.99	7.55
		(0,3)	10.02	9.54	9.06	8.56
			13.70	13.18	12.62	12.04
Clamped - free	I	(0,0)				
		(0,1)	13.49	12.93	12.35	11.75
		(0,2)	13.49	12.92	12.32	11.70
		(0,3)	14.11	13.56	12.97	12.36
			16.66	16.01	15.32	14.61

Table 5. Comparison of first four frequency parameter λ^2 of isotropic annular circular plate of parabolically varying thickness for different boundary conditions with different taper ratio using different methods.

BCS	Case	Mode	Non dimensional frequency parameter, λ^2			
			$T_x = 0.00$	$T_x = 0.25$	$T_x = 0.50$	$T_x = 0.75$
			NUM	NUM	NUM	NUM
Free- Free	parabolic	(0,0)	3.83	3.66	3.48	3.30
		(0,1)	8.82	8.45	8.07	7.68
		(0,2)	10.02	9.59	9.15	8.71
		(0,3)	13.70	13.23	12.74	12.22
Clamped - free	parabolic	(0,0)	13.49	12.99	12.47	11.93
		(0,1)	13.49	12.98	12.44	11.89
		(0,2)	14.11	13.61	13.09	12.55
		(0,3)	16.66	16.07	15.46	14.82

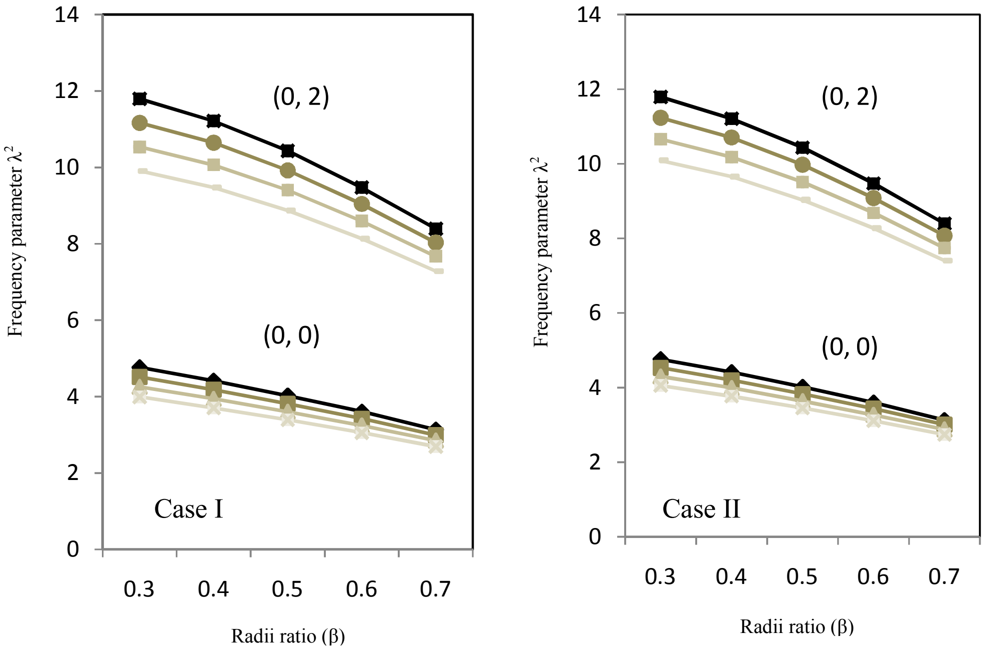


Fig. 3. Numerical comparison for the effect of (0, 0) and (0, 2) mode with different radii ratio (β) in terms of different taper ratio for (a) case I and (b) case II plate for free - free boundary conditions

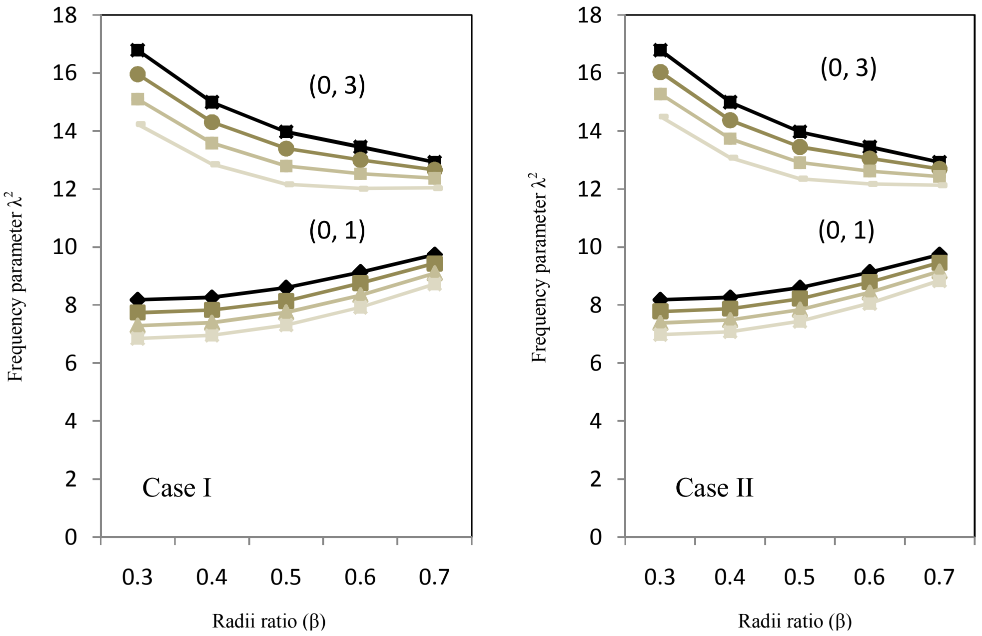


Fig. 4. Numerical comparison for the effect of (0, 1) and (0, 3) mode with different radii ratio (β) in terms of different taper ratio for (a) case I and (b) case II plate at free - free boundary conditions.

boundary conditions. It is clear from the Table 6 and Table 7 that the frequency parameters of Case III plate with linearly decreasing - increasing thickness variation reduce less in comparison to Case IV plate with parabolically decreasing - increasing thickness variation for different boundary conditions with different taper ratios. However, plate with Case V and Case VI plate with linearly and parabolically increasing – decreasing

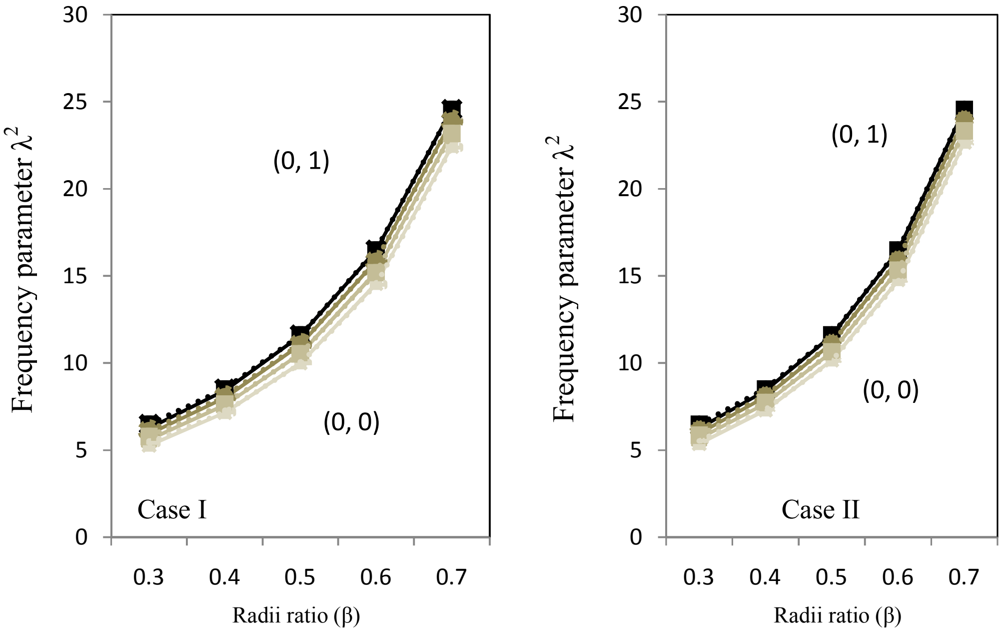


Fig. 5. Numerical comparison for the effect of (0, 0) and (0, 1) mode with different radii ratio (β) in terms of different taper ratio for (a) case I and (b) case II plate for clamped - free boundary conditions.

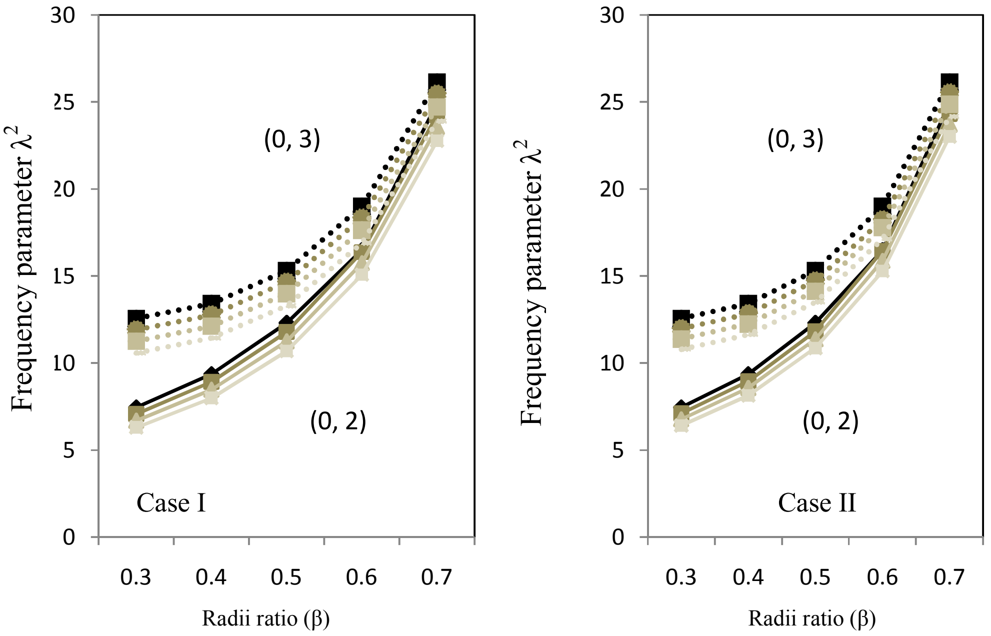


Fig. 6. Numerical comparison for the effect of (0, 0) and (0, 1) mode with different radii ratio (β) in terms of different taper ratio for (a) case I and (b) case II plate at clamped - free boundary conditions.

thickness variation does not have any significant effect on vibration response for different boundary conditions and for different taper ratios and behaves similar like a uniform plate. However, for all cases of combinations of arbitrarily varying thickness, plate with different linearly and parabolically varying thickness variation alters its modes at higher taper ratio. Therefore, it is observed that modes variation have significant impact on flexure response in comparison to the stiffness variation due to taper ratio**.** Hence it may be concluded that the Case V and Case VI plate will shows the highest resistance to flexural response than Case I and Case II plate due to mode variation. Further, modes with stiffness variation for plate with different cases of thickness variation provide a solution to flexural response. As for example, for flexural response actuation, Case I and Case II plate with linearly and parabolically decreasing thickness variation may be the option while for flexural response suppression, Case V and Case VI with linearly and parabolically increasing - decreasing thickness variation may be the another alternative solution.

Table 6. Numerical comparison of first four frequency parameter λ^2 of isotropic annular circular plate for different cases of thickness variation for free – free boundary conditions with different taper ratio.

Case	Mode	Natural frequency Parameter, λ^2			
		T_x= 0.00	T_x= 0.25	T_x= 0.50	T_x= 0.75
III	(0,0)	3.83	3.62	3.41	3.19
	(0,1)	8.82	8.37	7.91	7.43
	(0,2)	10.02	9.50	8.97	8.92
	(0,3)	13.70	13.13	12.52	11.88
IV	(0,0)	3.83	3.62	3.41	3.20
	(0,1)	8.82	8.38	7.93	7.46
	(0,2)	10.02	9.51	8.99	8.95
	(0,3)	13.70	13.14	12.55	11.92
V	(0,0)	3.83	3.83	3.82	3.82
	(0,1)	8.82	8.81	8.80	8.79
	(0,2)	10.02	10.01	10.00	9.49
	(0,3)	13.70	13.69	13.67	13.66

VI	(0,0)	3.83	3.83	3.82	3.83
	(0,1)	8.82	8.81	8.82	8.81
	(0,2)	10.02	10.02	10.02	10.01
	(0,3)	13.70	13.69	13.70	13.70

Table 7. Numerical comparison of first four frequency parameter λ^2 of isotropic annular circular plate for different cases of thickness variation for clamped – free boundary conditions with different taper ratio.

Case	Mode	Natural frequency Parameter, λ^2			
		T_x= 0.00	T_x= 0.25	T_x= 0.50	T_x= 0.75
III	(0,0)	13.49	12.87	12.21	11.53
	(0,1)	13.49	12.88	12.25	11.58
	(0,2)	14.11	13.51	12.87	12.20
	(0,3)	16.66	15.95	15.20	14.41
IV	(0,0)	13.49	12.88	12.24	11.51
	(0,1)	13.49	12.89	12.27	11.56
	(0,2)	14.11	13.52	12.89	12.17
	(0,3)	16.66	15.96	15.23	14.39
V	(0,0)	13.49	13.48	13.46	14.41
	(0,1)	13.49	13.48	13.46	13.45
	(0,2)	14.11	14.10	14.09	13.45
	(0,3)	16.66	16.65	16.63	14.08
VI	(0,0)	13.49	13.49	13.48	13.49
	(0,1)	13.49	13.49	13.49	13.48
	(0,2)	14.11	14.11	14.11	14.12
	(0,3)	16.66	16.66	16.65	16.66

IV. CONCLUSION

A comparison of axisymmetric vibration of a tapered annular circular plate having different linearly and parabolically varying thickness with different taper ratios for free- free and clamped - free boundary conditions is analyzed by keeping the total mass of the plate constant. It is observed that modes variation have significant impact on flexure response and thus the taper ratio causing the stiffness variation has the limited effect. Further, for taper ratio, $T_x = 0$, it is depicted that the effect of natural frequency parameter due to different combination of arbitrarily varying thickness is almost same as compared to that of the uniform plate. However, for increasing taper ratio, stiffness variation increases for different modes for different combination of arbitrarily varying thickness It is needless to mention that the non dimensional frequency parameter of Case I plate (linearly decreasing thickness variation) reduce more in comparison than that of Case II plate parabolically decreasing thickness variation) for both free – free and clamped – free boundary conditions. This is due to the less stiffness associated with the Case I plate than that of Case II plate. However, the effect of frequency parameter for Case III, Case IV, Case V and Case VI does not have any significant effect on natural frequency parameter and out of these plate, the natural frequency parameters for Case V plate (linearly increasing - decreasing thickness variation) and Case VI plate (parabolically increasing - decreasing thickness variation) are reported almost same as that of uniform plate due to higher stiffness associated with the plates. Further, modes with stiffness variation for plate with different cases of thickness variation provide a solution to flexural response. As for example, for flexural response actuation, Case I and Case II plate with linearly and parabolically decreasing thickness variation may be the option while for flexural response suppression, Case V and Case VI with linearly and parabolically increasing - decreasing thickness variation may be the another alternative solution.

REFERENCES

[1] C. M. Wang, G. M. Hong and T. J. Tan. Elasting buckling of tapered circular plates. computers& structure 55 , 1055-1061, 1995.

[2] A. P. Gupta, N. Goyal. Forced asymmetric response of linearly tapered circular plates. Journal of sound and vibration 220(4), 641-657, 1999.

[3] P. A. A. Laura, R. H. Gutierrez, V. Sonzognit and S. Idelsohn. Buckling of circular, annular plates of non uniform thickness. Ocean Engg , 24,51-61, 1997.

[4] Seema Sharma, R. Lal and Neelam. Free transverse vibrations of non-homogeneous circular plates of linearly varying thickness. Journal of international academy of physical sciences, 15, 187-200, 2011.

[5] C.Y. Wang. The vibration modes of concentrically supported free circular plates. Journal of sound and vibration 333, 835-847, 2014.

[6] Bani Singh and Saleh M. Hassan. Transverse vibration of a circular plate with arbitrary thickness variation. International journal of Mechanical Science, 40,1089-1104, 1998.

[7]W.H. Duana, C.M. Wang and C.Y. Wang. Modification of fundamental vibration modes of circular plates with free edges. Journal of Sound and Vibration, 317, 709–715, 2008.

[8] U.S Gupta, R. Lal, and Seema Sharma, Vibration of non-homogeneous circular Mindlin plates with variable thickness. Journal of Sound and Vibration, 302, 1-17, 2007.

[9] Jae-Hoon Kang, Three-dimensional vibration analysis of thick, circular and annular plates with nonlinear thickness variation. Computers &structure 81 , 1663-1675, 2003.

[10]A.W Leissa. Vibrations of Plates, Acoustical Society of America, New York, 1993.

[11]R.D.Mindlin and H.Deresiewich. Thickness-shear and flexural vibration of a circular disk, Journal of Applied Physics 25 (10), 1329–1332, 1954.

[12]O.G.Mcgee, C.S Huang and A.W Leissa. Comprehensive exact solutions for free vibrations of thick annular sectorial plates with simply supported radial edge, International Journal of Mechanical Science 37 (5) , 537–566, 1995.

[13]Hyeongill Lee, Rajendra Singh. Acoustic radiation from out-of-plane modes of an annular disk using thin and thick plate theories, Journal of Sound and Vibration 282, 313–339, 2005.

[14]T.Irie, G.Yamada, S.Aomura. Free vibration of a Mindlin annular plate of varying thickness, Journal of Sound and Vibration 66 (1), 187–197.

[15]C.Prasad, R.K Jain, S.R.Soni. Axisymmetric vibrations of circular plates of linearly varying thickness , Zamp 23, 941, 1972.

[16]L. E.. Luisoni, P.A.A.Laura and R. Grossi. Antisymmetric modes of vibration of a circular plate elastically restrained against rotation and of linearly varying thickness, Journal of Sound and Vibration 55 , 461–466, 1977.

[17]R. Grossi and P.A.A.Laura. Transverse vibrations of circular plates of linearly varying thickness, Applied acoustic, 13 , 7-18, 1980.

[18]B. Singh and V. Saxena. Axisymmetric vibration of a circular plate with double linear variable thickness, Journal of Sound and Vibration 179 , 879–897, 1995.

[19]P. A. A. Laura and B. Valerga De Greco. A note on vibrations and elastic stability of circular plates with thickness varying in bilinear fashion, Journal of Sound and Vibration 79 , 306–310, 1981.

[20]B. Singh and S. Chakraverty. Transverse vibration of a circular and elliptic plates with quadratically varying thickness, Applied Mathematical Modeling, 16 , 269-274, 1992.

[21]R .Barakat and E. Baumann. Axisymmetric vibrations of a thin circular plate having parabolic thickness variation, Journal of the Acoustical Society of America, 44 , 641-643, 1968.

[22]T.A. Lenox and H. D. Conway. An exact, closed form, solution for the flexural vibration of a thin annular plate having a parabolic thickness variation , Journal of Sound and Vibration, 68 , 231-239, 1980.

[23]D. R. Avalos, H.A. Larrondo and P. A. A. Laura. Transverse vibrations of a circular plate carrying an elastically mounted mass, Journal of Sound and Vibration 177, 251–258, 1994.

[24]U. S. Gupta, R. Lal and C.P. Verma. Buckling and vibration of polar orthotropic annular plates of variable thickness, Journal of Sound and Vibration 104 , 357–369, 1986.

[25]D.Y. Chen. Axisymmetric vibration of circular and annular plates with arbitrarily varying thickness, Journal of Sound and Vibration 206(1) , 114–121, 1997.

[26]D.G. Gorman. Natural frequencies of transverse vibration of polar orthotropic variable thickness annular plate, Journal of Sound and Vibration 86(1), 47–60, 1983.