PARAMETER ESTIMATION

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Abstract

Estimation is one of the core topics in mathematical statistics. It refers to the process by which one makes inferences about a population, based on information obtained from a sample. The aim of the present paper is to focus on estimation process that is expressed in two ways: Point estimation, Interval estimation and to differentiate between two. We study the most common method of point estimation: the method of moments, the method of maximum likelihood, and Bayes' estimators, properties of good estimator and sufficient conditions for an estimator to be the best possible one. We also study sample mean and standard deviation as the most popular point estimates of population mean and standard deviation and find confidence intervals for mean and error in estimation.

Key Words: Estimation, Point estimation, Interval estimation, Confidence Intervals

INTRODUCTION

The one of the important area in mathematical statistics is estimation of population parameters and setting up of confidence interval for them. It refers to the process by which we make inferences about a population on the basis of information obtained from given sample. The entire purpose of estimation theory is to arrive at an estimator. Single value of estimator alone is not enough for a complete inference; we also need measure of uncertainty. The confidence interval for estimator gives us range around a point estimate and also certainty in terms of the probability that an interval contains the population parameter in this range.

DEFINITION AND NOTATION

Random Experiment and trial

An experiment, the outcome of which cannot be predicted with certainty, but every possible outcome can be described prior to its performance when such experiment is repeated under the same condition is called *random experiment***.** Each performance in random experiment is called *trial*. [10]

Sample space

The sample space **or** universal sample space**,** the collection of every possible outcome of an random experiment is called the experimental space or *sample space* often denoted *S*, Ω.

Random variable

Def : Given a random experiment with sample space *S*. A function X which assign to each element c in *S*. one and only one X(c)=x, is called *Random variable.* The space of X Is the set of real number A={x;x=X (c),c in S }.[10]

Target population

Def: The totality of elements is under discussion and about which information is desired Will be called the target population. [7]

Random Sample

Def : Let the random variable $X_1, X_2, \ldots X_n$ have joint density

$$g(x_1, x_2 \ldots x_2) = f(x_1)\, f(x_2) \ldots\ldots\ldots (x_n)$$

where density of each X_i is f(x). then $X_1, X_2 \ldots\ldots X_n$ is said to be random sample of size n from population with density, f(x).[7]

Sample Population

Def: Let X_1, X_2..........X_n be random sample from a population with density f(x); then this population is called sampled population.

Statistic

Def: Statistic is a function of observable random variable which does not contain any unknown parameter. [7]

Unbiased estimator

Def: An estimator U_n of parameter θ is said to be unbiased If there exist the relation

$$E(U_n)= \theta$$

Consistent estimator

An estimator U_n of the parameter θ is called consistent for the parameter θ if the sequence $\{U_n\}$ converges stochastically to the parametric value of θ

Point estimator

The estimator is a function of random variables and takes the measured data as input and generates an estimate of the parameter. Estimator optimality is an important point under consideration. We try to determine best estimate of θ. Firstly we understand the meaning of optimal estimator of θ. In order to qualify an estimator $\hat{\theta} = u(X_1, X_2, \ldots\ldots X_n)$ as good point estimator of θ. There should be great probability that $\hat{\theta}$ will be close to θ. It can be achieved by selecting $\hat{\theta} = u(X_1,X_2,\ldots\ldots,X_n)$ in such a way that not only is $\hat{\theta}$ an unbiased estimator of θ but also variance of $\hat{\theta}$ is as small as possible. Here we have enlisted some properties of Good Point Estimator:-

Unbiased Estimators

An $\hat{\theta}$ is a good estimator of θ if means squared error will be as small as possible.

Now means squared error is

$$E[(\hat{\theta} - \theta)^2] = Var\text{-}\hat{\theta} + [\theta - E(\hat{\theta})]$$

It contains two terms. Term $[\theta - E(\hat{\theta})]$ is called biased and it must be tends to zero and second term Var-$\hat{\theta}$ is as small as possible for good estimation.

Thus term $[\theta - E(\hat{\theta})]$ must be equal to zero

$$E(\hat{\theta}) = \theta$$

Which in turn implies $\hat{\theta}$ must be *unbiased.*

1. Uniformly Minimum Variance Unbiased Estimator (UMVUE)

We need unbiased estimator with minimum variance . It is possible to find an estimator $\hat{\theta}$ which has minimum variance among all other estimators for all values of $\theta \in \Theta$. In basic statistical model in which random experiment results in observable random variable X taking values in S (sample space) X is of the form λ

$X = (X_1, X_2, \ldots, X_n)$; where X_i is the vector of measurement for the ith item.

Let θ is the real parameter of distribution of X, taking value in parameter space Θ. X has p.d.f. f(x,θ) here expected value variance covariance of X also depends on θ. Let $\lambda = \lambda(\theta)$ is parameter of our interest that is derived fromθ. (It may happen $\theta = \lambda(\theta)$). If U and V are unbiased estimator of λ and

$$Var(U) \leq var(V) \quad \text{for all } \theta \in \Theta$$

U is uniformly better estimator than V.

If U is uniformly better than any unbiased estimator of λ, then U is a *Uniformly Minimum Variance Unbiased Estimator (UMVUE)* of λ. Under milder condition it can be shown that there is a lower bound of any unbiased estimator of the parameter λ. This ensures that we can find out an estimator that achieves lower bound for all $\theta \epsilon \Theta$ that means we can find an estimator which is UMVUE of λ.

Consistent Estimator

A good estimator should be one for which risk become small as sample size increases. Mathematically $\hat{\Theta}_1$, $\hat{\Theta}_2$,…..,$\hat{\Theta}_n$ are sequence $\{\hat{\Theta}_n\}$ of estimators of θ. Then this sequence is a simple consistent estimator of θ if $\{\hat{\Theta}_n\}$ converges to θ.

Sufficient Statistic

On the basis random sample of n-values $X_1, X_2,……….., X_n$ sample statistic $\hat{\theta} = d(x_1, x_2,…………, x_n)$ used to estimate population parameter .since statistic $\hat{\theta}$ is a random variable taking $x_1, x_2,…………, x_n$ as its input and condensed to a single random variable .then it may happen that we may lost some information in this condensing process . Some choice of statistic say $d(x_1, x_2,…………, x_n) = x_1$ may not utilize all the information in the sample. For an estimator $\hat{\theta}$, It is good to contain all information about parameter θ that is lies in sample value. If such $\hat{\theta}$ exist is called sufficient statistics.(e.g sample mean utilize all the information in the sample) Mathematically it is possible if for any other estimator θ^* conditional probability density of θ^* given $\hat{\theta}$ ($p(\theta^*/\hat{\theta})$) does not involve θ.

An estimator has one or more properties of good estimator e.g. sample mean as an estimator of population mean has all properties of good estimator enlisted above. Hence sample mean is an efficient estimator even asymptotically efficient estimator. However, optimal estimators do not always exist. [7]

Point estimation /interval estimation

Central problem in statistic is to study a population which has a density $f(x; \theta)$ where parameter θ is unknown but form of density is known. We take random sample of $X_1, X_2,……., X_n$ of size n from this density. Let some statistic $u(X_1, X_2, ……………………… X_n)$ estimate unknown parameter θ. A point estimate of a population parameter is a single value of a statistic the sample mean μ_s is a point estimate of the population mean μ and the sample proportion p is a point estimate of the population proportion *P*. An interval estimate is defined by two numbers, between which a population parameter is said to lie. For example, $\alpha < \mu < \beta$ is an interval estimate of the population mean μ. It indicates that the population mean is greater than *α* but less than *β*. Confidence interval express the precision and uncertainty associated with a particular sampling method. To describe a confidence interval we need to define a *statistic, confidence level, a margin of error.* Confidence intervals are preferred to point estimates, because confidence intervals indicate the precision of the estimate and the uncertainty of the estimate.

METHOD OF CONSTRUCTION OF POINT ESTIMATOR

Method of Maximum Likelihood Estimator (MLE)

Find frequency distribution density $f(x; \theta)$ for population. consider a random sample $X_1, X_2, \ldots X_n$ from distribution which having p.d.f. $f(x; \theta)$ where θ lies in a parameter space Θ . Define likelihood function L on random sample $X_1, X_2, \ldots X_n$ as their joint p.d.f

$$L(\theta; x_1, x_2, \ldots x_n) = f(x_1, \theta)\, f(x_2, \theta) \ldots\ldots f(x_n, \theta)$$

Now we can find a non trivial function $u(x_1, x_2 \ldots x_n) = \hat{\theta}$ such that when we replace θ by the $u(x_1, x_2 \ldots x_n)$ value of L is maximum. (Consider L as a function of θ and maximize it with the help of derivative technique. Find $\hat{\theta}$ as a point of maxima.) Then the statistic $\hat{\theta}$ of will be called maximum likelihood estimator of θ. Sample mean is maximum likelihood estimator for population mean. Maximum likelihood estimators are easy to obtain since L (θ) and log L (θ) has their maximum at same value of θ, and it is easier to find maximum of logarithm likelihood function.

It is important to note that it is not always rely on differentiation process to locate maximum likelihood estimator. Another point is to note is under quite general regularity condition on density function, f(x,θ) maximum likelihood estimator have some optimum properties. It can be shown that among all estimates, MLE are asymptotically normally distributed the maximum likelihood estimator possesses minimum asymptotic variance, hence can be considered as best among all asymptotically normally distributed estimates. But maximum likelihood estimator are often biased .For unbiased maximum likelihood estimator, we can multiply estimate by a constant that depends on n, such as was done with s^2 (sample variance). In some problem, it is not possible to adjust maximum likelihood estimator in this manner .

Method of Moments (MOM)

The oldest method is the method of moments which use K-th sample moment as an estimate of K-th moment of population parameter. The most serious shortcoming of MOM estimator as compare to MLE and Bayes Estimators is that it may not based on sufficient statistics. Simply saying that MOM estimators are inefficient in using all the information about population parameter that is in the sample. In MLE realization of observed sample is as large as possible. Bayes estimators are also based on sufficient statistics. In MOM estimators are found by equating first k sample moments to corresponding population moments. [10]

Bayes Estimator

Bayesian analysis is named after famous scientist for Thomas Bayes, Here the parameter treated as θ as random variable, with a given probability density function $h(\theta)$ for $\theta \in \Theta$. The corresponding distribution of θ is called *prior distribution* of θ is intended to reflect knowledge of parameter before we gather data. After observing X belongs to S ; S is sample space . we use BAYES Theorem to compute conditional probability density function $h(\theta/x)$ which gives us conditional probability given X=x. The conditional distribution of θ given X=x is called posterior distribution and update given distribution information in the data.

Finally the conditional expected value $E(\theta / X)$ is the bayes estimator of θ and the $E(\theta / X)$ is a function of X and among all functions of X is closest to θ in the mean square sense.

Interval Estimation

Point estimation is not 100% accurate. A single value cannot give complete information regarding actual parameter. If sample is sufficiently large and estimator is maximum likelihood estimator we can use normal curve method to find an interval for estimation which are called confidence interval for estimation of population parameter. It involves a range and a probability. If μ_s and σ_s are mean and standard deviation of sampling distribution of a statistic S (maximum likelihood estimator), Than probability of finding an actual sample statistic in the interval (μ_s - σ_s to μ_s + σ_s), (μ_s - 2σ_s to μ_s + 2 σ_s), (μ_s - 3σ_s to μ_s + 3 σ_s) is 68.27%, 95.45%, 99.73% respectively. we call respective interval the 68.27%, 95.45%, 99.73% confidence interval for estimating μ_s. For the simplicity, we make 95% and 99% confidence interval. S$\pm$ 1.96 σ_s ___ are confidence limit for 95% and 99% confidence interval is S$\pm$ 1.96 σ_s, S$\pm$ 2.58 σ_s respectively.

Confidence Interval for Mean

In most of problem, sample mean μ_s is maximum likelihood estimator for estimating population mean. So it is good to find out confidence limit for μ_s. more generally confidence limits are given by μ_s.$\pm$ z_c σ_s ;where Z_c is called *confidence coefficient* depend on particular levels. 95% and 99% confidence level, the value of Z_c are 1.96 and 2.58 for 95% and 99% confidence level respectively. Also term *Error probability* **(**α**)** is used to describe Confidence Interval .It is defined as 1- Z_c .

CONCLUSION

From the above, we may infer that maximum likelihood method for constructing estimator is most popular amongst all other methods determining the point estimator and its popularity is justified because it provides an easy method for constructing estimators on a population parameter and estimator qualify optimal condition. Only sometime, maximum likelihood estimator are biased e.g. sample variance is biased estimator of population variance but this can be made unbiased by multiplying a numerical factor with sample variance (n/n-1); n being the size of random sample. However, a point estimator gives single value to population parameter which is alone not enough for complete inference. Also we need a measure of uncertainty. We need a set of estimator *C*. In case of real-valued parameter, we usually prefer the set estimate *C* to be an interval. And uncertainty in estimation is quantified by the size of the interval and its probability of covering the parameter θ. These confidence interval provide range around a point estimate and certainty in terms of the probability that an interval contains the population parameter between its lower bound and upper bound.

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