Fixed Points of Geraghty contractions in rectangular metric spaces

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ABSTRACT: In this paper, we define generalized Geraghty contraction with rational type maps in rectangular metric spaces with α, η admissible function and an altering distance function φ, and prove the existence of fixed points. Our results extend the some of the known results.
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1. Introduction and Preliminaries

Banach contraction principle is one of the fundamental results in fixed point theory . There are several generalizations of metric spaces . In 2000 Braniciari generalized metric spaces , in which triangular inequality is replaced by quadrilateral inequality which is known as rectangular metric spaces.

1. Rectangular metric spaces need not be continuous.
2. In rectangular metric space, a convergent sequence need not be a Cauchy sequence.
3. Rectangular metric spaces need not be a Haussdorff space.

Definition 1.1 [3] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ satisfy the following conditions for all x, y $\in$ X and all distinct u, v $\in$ X each of them different from x and y

(i) $d(x, y) = 0$ if and only if x=y
(ii) $d(x, y) = d(y, x)$, and
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. (quadrilateral inequality)

Then the function d is called a rectangular metric and the pair (X, d) is called a rectangular metric space (in short RMS).

Definition 1.2 [3] Let (X, d) be a rectangular metric space (in short RMS) and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is called (g.m. s) convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is called (g.m. s) Cauchy sequence if and only if for every $\epsilon > 0$ there exists positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $m > n \geq N(\epsilon)$.

(iv) A rectangular metric space (X, d) is called complete if every (g.m.s) Cauchy sequence is a (g. m.s) convergent.

Definition 1.3 ([12]) A function : $R^+ \to R^+$, $R^+ = [0, \infty)$ is said to be an *altering distance function* if the following conditions hold:

(i) φ is continuous,

(ii) φ is non-decreasing, and

(iii) $\varphi(t) = 0$ if and only if $t = 0$.

In 1973, Geraghty [8] introduced a new contractive mapping in which the contraction constant was replaced by a function having some specific properties taken from the class of functions S, where $S = \{\beta : [0, \infty) \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$

Definition 1.4 . [13] Let T: XxX be a self map and α: XxX $\to R$ be a function . Then T is said to be α – admissible function if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 1.5 . [13] Let T: XxX be a self map on a metric space (X, d) and α, η : XxX $\to [0, \infty)$ be two functions . Then T is said to be α – admissible mapping with respect to η if $\alpha(x, y) \geq \eta(x,y)$ implies $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$ for all x, y ∈ X.

If $\eta(x, y) = 1$ for all x, y ∈ X , then T is called α- admissible mapping.

Definition 1.6. Let (X, d) be a rectangular metric space and let T: X → X be a self map. If there exists β ∈ S such that $d(Tx, Ty) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$

Where $M(x, y) = \max\{ d(,y), d(x,Tx), d(y,Ty), \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty), \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty) \}$

for all x , y ∈ X then we call T is a φ_M - generalized Geraghty contraction in rectangular metric spaces.

Lemma 1.10. [2] Let (X, d) be metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. For each $k > 0$, corresponding to m(k), we can choose n(k) to be the smallest integer such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)\ 1}) < \epsilon$. It can be shown that the following identities are satisfied.

(i) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$ (*ii*) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$,

(*iii*) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon$, and (*iv*) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$.

Now, we prove the existence of fixed points of generalized Geraghty contraction maps with rectangular metric spaces .

2. MAIN RESULTS

Theorem 2.1. Let (X, d) be a Hausdorff and complete rectangular metric space. Let $T: X \to X$ be an α- admissible mapping with respect to η. Assume that there exists an altering distance function φ *such that $x, y \in X$,*

$\alpha(x, y) \geq \eta(x,y)$, implies $d(x,y) \leq \beta(\varphi(M(x,y)))\varphi(M(x,y))$ 2.1.1

where $M(x,y) = \max\{ d(,y), d(x,Tx), d(y,Ty), \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty), \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty) \}$

Also, suppose that the following assertions are hold; Geraghty contraction. Suppose that

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$

(ii) for all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $\alpha(x, z) \geq \eta(x, z)$

(iii) T is continuous.

Then T has a periodic point $a \in X$ and $\alpha(x, Ta) \geq \eta(a, Ta)$ holds for each periodic point then T has a fixed point.

Proof. By (1) , there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. *2.1.2*

We define $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for n=1,2,3,… .

If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence x_n is a fixed point of T. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in N$.

Since T is α admissible mapping with respect to η and consider 2.1.2

we have

$\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0)$

$\geq \eta(Tx_0, T^2x_0) = \eta(x_1, x_2)$.

By mathematical induction,

it is easy to see that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N$.

We consider $d(x_n, x_{n+1}) = \varphi(d(Tx_n - 1, Tx_n))$

$$\leq \alpha(x_n, x_{n+1})\, \varphi(Tx_n, Tx_{n+1})$$
$$\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(M(x_{n-1}, x_n)) \qquad (2.1.3)$$

Now

$$M(x_{n-1}, x_n) = \max\{ d(x_{n-1}\, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{1+d(x_{n-1}, x_n)}[d(x_{n-1}, x_n)d(x_n, x_{n+1})], \frac{1}{1+d(x_n, x_{n+1})}[d(x_{n-1}, x_n)d(x_n, x_{n+1})]\}$$

$$\leq \max|\{ d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{1+d(x_{n-1},x_n)}[d(x_{n-1},x_n)d(x_n,x_{n+1})], \frac{1}{1+d(x_n,x_{n+1})}[d(x_{n-1},x_n)d(x_n,x_{n+1})]\}$$
$$= \max|\{ d(x_{n-1},x_n), d(x_n,x_{n+1}) \} \quad .$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then from (2.1.1), we have

$$\varphi(d(x_n, x_{n+1})) \leq \beta(\varphi(M(x_{n-1}, x_n)))(\varphi(M(x_{n-1}, x_n))$$
$$\leq \beta(\varphi(M(x_{n-1}, x_n)))(\varphi(d(x_{n-1}, x_n))$$
$$< \varphi(d(x_n, x_{n+1})), \text{ a contradiction.}$$

So that we have $\max\{ d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$, and hence

$$\varphi(d(x_n, x_{n+1})) \leq \beta(\varphi(M(x_{n-1}, x_n)))(\varphi(M(x_{n-1}, x_n)) < \varphi(d(x_{n-1}, x_n))$$ for all n.

Thus it follows that $\{ \varphi(d(x_n, x_{n+1})) \}$ is a decreasing sequence of non negative reals and so $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1}))$ exists and it is r(say). i.e., $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r \geq 0$.

We now show that $r = 0$.

If $r > 0$ then from 2.1.3

$$\begin{aligned}\varphi\big(d(x_n, x_{n+1})\big) &= \varphi\big(d(Tx_{n-1}, Tx_n)\big)\\ &\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(M(x_{n-1}, x_n))\\ &\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(d(x_{n-1}, x_n)),\end{aligned}$$ and hence

$$\frac{\varphi\big(d(x_n\ ,\ x_{n+1})\big)}{\varphi(d(x_{n-1}, x_n)} \leq \beta(\varphi(M(x_{n-1}, x_n))) < 1\ for\ each\ n \geq 1.$$

Now on letting $n \to \infty, we\ get$

$$1 = \lim_{n\to\infty} \frac{\varphi\big(d(x_n, x_{n+1})\big)}{\varphi\sigma(x_{n-1}, x_n)} \leq \lim_{n\to\infty} \beta(\varphi(M(x_{n-1}, x_n))) \leq 1$$

So that $\beta(\varphi(M(x_{n-1}, x_n))) \to 1\ as\ n \to \infty$.

This implies that $\lim_{n\to\infty}(\varphi(M(x_{n-1}, x_n))) = 0$.

Since $\varphi\big(d(x_{n-1}, x_n)\big) \leq \varphi\big(M(x_{n-1}, x_n)\big)\ for\ all\ n, we\ have$

$$\lim_{n\to\infty}\Big(\varphi\big(d(x_n, x_{n+1})\big)\Big) \leq \lim_{n\to\infty}\Big(\varphi\big(M(x_{n-1}, x_n)\big)\Big) = 0.$$

Hence $\lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0. i.e., r = 0$.

Now we prove that $\varphi(d(x_n,\ x_{n+2}) \to 0$ as $n \to \infty$.

$$\begin{aligned}\varphi(d(x_n,\ x_{n+2}) &= \varphi(\ d(Tx_{n-1},\ Tx_{n+1}))\\ &\leq \varphi(Tx_{n-1}, Tx_{n+1})\\ &\leq \beta(\varphi(\ M(x_{n-1}, x_{n+1})))\varphi(\ M(x_{n-1}, x_{n+1}))\end{aligned} \qquad (2.1.4)$$

Now

$$M(x_{n-1}, x_{n+1}) = \max\{ d(x_{n-1}\ x_{n+1}),\ d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}),\ \frac{1}{1+d(x_{n-1}, x_{n+1})}[d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})],\ \frac{1}{1+d(x_n, x_{n+2})}[d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})]\}$$

$$\leq \max|\{ d(x_{n-1}, x_{n+1}),\ d(x_{n-1}, x_n),\ d(x_{n+1}, x_{n+2}), [d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})], [d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})]\}$$

$$= \max|\{ d(x_{n-1}, x_n),\ d(x_n, x_{n+1}) \}\quad .$$

Since $\varphi(d(x_n, x_{n+1})) < \varphi\ (d(x_{n-1}, x_n))$ it follows that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$

Therefore $M(x_{n-1}, x_{n+1}) \leq \max\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), [d(x_{n-1}, x_n)]^2, [d(x_{n-1}, x_n)]^2\}$

$$\leq \max\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), [d(x_{n-1}, x_n)]^2\}$$

Let $a_n = d(x_{n-1}, x_{n+1})$ and $b = d(x_{n-1}, x_n)$.

Thus $M(x_{n-1}\ x_{n+1}) = \max\{a_n,\ b_n, [b_n]^2\}$ for each $n \geq N$.

Here we have three cases. If $M(x_{n-1}\ x_{n+1}) \leq b_n$ or $M(x_{n-1}\ x_{n+1}) \leq [b_n]^2$.

Since $b_n \to 0$ as $n \to \infty$ from 2.1.4

$$\lim_{n\to\infty} \varphi(d(x_n,\ x_{n+2}) \leq \lim_{n\to\infty} \beta(\varphi(M(x_{n-1},\ x_{n+1})))\varphi(M(x_{n-1},\ x_{n+1})) = 0.$$

If $M(x_{n-1}\ x_{n+1}) \leq a_n$, then we see that

$$\varphi(d(x_n,\ x_{n+2}) \leq \beta(\varphi(M(x_{n-1},\ x_{n+1})))\varphi(M(x_{n-1},\ x_{n+1})) < \varphi(d(x_{n-1},\ x_{n+1}).$$

Thus, the sequence $\{ d(x_n,\ x_{n+2}) \}$ is a decreasing sequence of non-negative real numbers and hence $d(x_n,\ x_{n+2}) \to 0$ as $n \to \infty$. 2.1.5

Now, we claim that T has a periodic point. Assume that T has no periodic point, then $\{x_n\}$ is a sequence of distinct points, that is $x_n \neq x_m$ for all $m \neq n$. In this case we will get that $\{x_n\}$ is a g. m. s Cauchy sequence. If not, then there exists ϵ >0 for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ $m(k) > n(k) > k$ for each k ≥ N , such that

$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$ and $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$

$\{x_n\}$ is a sequence of distinct points, then from rectangular inequality , we have

$$\begin{aligned}\varepsilon &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq d(x_{m(k)}, x_{n(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k-1)}, x_{n(k)})\\ &\leq d(x_{m(k)}, x_{n(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + \varepsilon.\end{aligned}$$

Thus, $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon.$

Using rectangular inequality we have $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.$

From 2.1.4 $\varepsilon \leq d(x_{m(k)}, x_{n(k)})$

$$\leq \beta(\varphi(M(x_{m(k)-1}, x_{n(K)-1})))\,\varphi(M(x_{m(k)-1}, x_{n(K)-1}))$$

Where

$$M(x_{m(k)-1}, x_{n(K)-1}) = \max\{ d(x_{m(k)-1}, x_{n(k)-1}),\ d(x_{m(k)-1}, x_{m(k)}),\ d(x_{n(k)-1}, x_{n(k)}).\ \frac{1}{1+d(x_{m(k)-1}, x_{n(k)-1})}[d(x_{m(k)-1}, x_{nmk)})\,d(x_{n(k)-1}, x_{n(k)})],\ \frac{1}{1+d(x_{m(k)}, x_{n(k)1})}[d(x_{m(k)-1}, x_{nmk)})\,d(x_{n(k)-1}, x_{n(k)})]\}$$

On letting $k \to \infty$

$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(K)-1}) = \varepsilon.$

Now, we have

$$\begin{aligned}\varphi\left(d\left(x_{m(k),}\, x_{n(k)}\right)\right) &\leq \beta\left(\varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right)\right)\varphi(M(x_{m(k)-1,}\, x_{n(k)-1}))\\ &\leq \beta\left(\varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right)\right)\varphi(M\left(x_{m(k-1),}\, x_{n(k)-1}\right))\\ &\leq \beta\left(\varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right)\right)\varphi(d\left(x_{m(k-1),}\, x_{n(k)-1}\right))\end{aligned}$$

And hence

$$\frac{\varphi(d\left(x_{m(k),}\, x_{n(k)}\right))}{\varphi(d\left(x_{m(k)-1,}\, x_{n(k)-1}\right)} \leq \beta\left(\varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right)\right) < 1.$$

On letting $k \to \infty$ and from the Lemma 1.11, we get

$$1 = \frac{\varphi(\epsilon)}{\varphi(\epsilon)} \leq \lim_{k\to\infty} \beta(\varphi(M(x_{m(k-1),}\, x_{n(k)-1}))) \leq 1$$

So that $\beta\left(\varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right)\right) \to 1\ as\ k\ \to \infty.$

Since $\beta \in S, \varphi\left(M\left(x_{m(k-1),}\, x_{n(k)-1}\right)\right) \to 0\ as\ k \to \infty. i.e., \varphi(\epsilon) = 0,$

Since φ is continuous. Hence it follows that $\epsilon = 0$, a contradiction.

Therefore $\{x_n\}$ is a Cauchy g.m.s. sequence in X, and since (X,d) is complete, there exists $z \in X$ such that $\{x_n\}$ is g.m.s convergent to z,

Now, we show that z is a fixed point of T.
First we assume that (iii) hold. i.e., T is continuous.
$x_{n+1} = Tx_n \to Tz$ as $n \to \infty$
and since X is Hausdorff we have z= Tz.
Therefore z is a fixed point of T in X.

Theorem 2.2. Let (X, d) be a Hausdorff and complete rectangular metric space. Let T: $X \to X$ be an α- admissible mapping with respect to η. Assume that there exists an altering distance function φ *such that $x, y \in X$,*
α (x, y) $\geq \eta$(x,y) , implies $d(x,y) \leq \beta(\varphi(M(x,y)))\varphi(M(x,y))$ 2.2.1
where M(x,y) = max{ $d(,y),\ d(x,Tx),\ d(y,Ty), \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty),\ \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty)$ }
Also, suppose that the following assertions are hold. Suppose that

(ii) there exists $x_0 \in X$ such that α (x_0, Tx_0) ≥ η (x_0, Tx_0)
(ii) for all x, y $\in X$, α (x, y) ≥ η (x, y) and α (y, z) ≥ η (y, z) implies α (x, z) ≥ η (x, z)
(iii) X is α-regular with respect to η .

Then T has a periodic point a $\in X$ and α (x, Ta) ≥ η (a, Ta) holds for each periodic point then T has a fixed point. Moreover, if for all x,y $\in F(T)$, we have α (x, y) ≥ η (x, y), then the fixed point is unique.

Proof. From the proof of the theorem2.1, we have the sequence $\{x_n\}$defined by $\{x_{n+1}\} = Tx_n$ for all $n \geq 0$is a Cauchy in (X, σ) and converges to to some $z \in X$.
Let X be α-regular with respect η
Also 2.1.3 we have $\alpha(x_n, z) \geq \eta(x_n, z)$, for all n≥N.
$\varphi(d(Tx_n, Tz)) \leq \beta(\varphi(M(x_n, z)))\varphi(M(x_n, z))$ (2.2.1)

Where
$M(x_n, z) = \max\{ d(x_n, z),\ d(x_n, x_{n+1}),\ d(z,Tz).\ \frac{1}{1+d(x_n, z)}[d(z,Tz)d(z,Tz)],\ \frac{1}{1+d(x_{n+1},Tz)}[d(z,Tz)d(x_n, x_{n(+1)})]\}$

since $\{x_n\} \to z$ as $n \to \infty$, then we have
$\lim_{k\to\infty} M(x_n, z) \leq d(z,Tz)$
Taking limit as $n \to \infty$, from 2.2.1 and using the continuity of φ we get
$\varphi(d(z,Tz)) \leq \beta(\varphi(M(x_n, z)))\varphi(M(x_n, z))$
Which implies that $\varphi(d(z,Tz))$ =0 implies that $d(z,Tz) = 0$ and so z= Tz.
Hence T has a periodic point,
Now we show that T has a fixed point.
There exists $a \in X$ such that $a = T^p a$. It is clear that $a \in X$ is a fixed point of T for p=1.
We will prove that $\upsilon = T^{p-1}a$ is a fixed point of T. In case of p>1. If possible, assume the contrary, i.e., let $T^{p-1}a \neq T^p a$.
As $\alpha(a, Ta) \geq \eta(a,Ta)$ and T is α admissible w.r.t η

we have $\alpha(T^n a,\ T^n Ta) \geq \eta\ (T^n a,\ T^n Ta)$ for all $n \in N$.

From 2.2.1, we have

$$\varphi(d(a,Ta)) = \varphi(d(T^p a, T^p Ta)) \leq \beta(\varphi(M(T^{p-1}a, T^p a)))\,\varphi(M(T^{p-1}a, T^p a))$$

Where

$$M(T^{p-1}a,\ T^p a) = \max\{\ d(T^{p-1}a,\ T^p a),\ d(T^{p-1}a,\ T^p a), d(T^p a,\ T^{p+1}a),\ \frac{1}{1+d(T^{p-1}a,\ T^p a)}[d(T^{p-1}a,\ T^p a)\,d(T^p a,\ T^{p+1}a)],\ \frac{1}{1+d(T^p a,\ T^{p+1}a)}[d(T^{p-1}a,\ T^p a)\,d(T^p a,\ T^{p+1}a)]\}$$

$$= \max\{d(T^{p-1}a,\ T^p a), d(T^p a,\ T^{p+1}a)\}$$

If $M(T^{p-1}a,\ T^p a) = d(T^p a,\ T^{p+1}a)$, then we get contradiction.

So that $M(T^{p-1}a,\ T^p a) = d(T^{p-1}a,\ T^p a)$

$$\varphi(d(a,Ta)) = \varphi(d(T^p a, T^p Ta)) \leq \beta(\varphi(M(T^{p-1}a,\ T^p a)))\,\varphi(d(T^{p-1}a,\ T^p a)) < \ d(T^{p-1}a,\ T^p a)$$

$$\varphi(d(a,Ta)) < \varphi(\ d(T^{p-1}a,\ T^p a)) < \varphi(\ d(T^{p-2}a,\ T^{p-1}a)) < \ .... < \varphi(d(a,\ Ta))$$

Since φ is continuous it follows that $\vartheta = T^{p-1}\vartheta$ is not a fixed point of T is not true.

Consequently, T has fixed point.

Now we shot that the fixed point is unique.

If possible, let $\vartheta, \vartheta^1 \in X$ be distinct fixed point of T. Then $\alpha(\vartheta, \vartheta^1) \geq \eta\ (\vartheta, \vartheta^1)$.

From the inequality 2.2.1, we have

$$\varphi(d(\vartheta, \vartheta^1)) = \varphi(d(T\vartheta, T\vartheta^1)) \leq \beta(\varphi(M(\vartheta, \vartheta^1)))\,\varphi(M(\vartheta, \vartheta^1)) \qquad (2.2.2)$$

$$M(\vartheta, \vartheta^1) = \max\{d(\vartheta, \vartheta^1),\ d(\vartheta, T\vartheta^1),\ d(\vartheta^1, T\vartheta^1),\ \frac{1}{1+d(\vartheta, \vartheta^1)}[d(\vartheta, T\vartheta)\,d(\vartheta', T\vartheta^1)],\ \frac{1}{1+d(T\vartheta, T\vartheta^1)}[d(\vartheta, T\vartheta)\,d(\vartheta', T\vartheta^1)]\}$$

$$= \ d(\vartheta, \vartheta^1) .$$

From(2.2.2) , we have

$$\varphi(d(\vartheta, \vartheta^1)) = \varphi(d(T\vartheta, T\vartheta^1)) \leq \beta(\varphi(M(\vartheta, \vartheta^1)))\,\varphi(M(\vartheta, \vartheta^1)) < \varphi(d(\vartheta, \vartheta^1))$$

Since φ is continuous it follows that $d(\vartheta, \vartheta^1)$ =0, i.e., fixed point is unique.

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