UNIFORM STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, the stability of general impulsive retarded functional differential equations where state variables on impulses are related to time delay has been considered. By using Lyapunov functions and analysis along with Razumikhin technique, some criteria for the uniform stability of impulsive differential equations have been derived. The obtained result extends and generalizes some results existing in the literature.

Keywords: **Impulsive functional differential equations, Uniform stability, Lyapunov function, Razumikhin technique.**

1. INTRODUCTION

A Impulsive differential equations have attracted many researchers because of their ample potential applications in many areas such as control technology, industrial robotics, drug administration and so on. Many classical findings have been extended to impulsive systems[4,6,8,9]. By Lyapunov's direct method, various stability problems have been discussed for impulsive delay differential equations[1,3] .In addition, there have been several research papers recently on stability analysis of delay differential equations[2,10,11]. The method of Lyapunov functions and Razumikhin technique have been widely applied to stability analysis of various delay differential equations [7]. Numerous research works are in the literature, on impulsive delayed linear differential equations. But not much is done in state variables related to time delay. In this paper we will study the uniform stability of impulsive differential equation in which state variables at the time of impulses are related to time delay. As a result criteria on uniform stability can be derived.

This paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we get some criteria for uniform stability of the system of the impulsive differential equations. Finally, concluding remarks are given in Section 4.

2.PRELIMINARIES

Consider the following functional impulsive differential:

$$\begin{cases} x'(t) = f(t, x_t), t \neq t_k, t \geq t_0 \\ \Delta x(t_k) = x(t_k^-) + I_k(x(t_k^-)) + J_k(x(t_k^- - \tau)), \quad k \in N \\ x(t + t_0) = \varphi(t), t \in [-\tau, 0] \end{cases} \quad (1)$$

We assume that function f,: $R_+ \times PC([-\tau, 0], R^n) \to R^n), \tau > 0$ and $\varphi \in PC([-\tau, 0], R^n)$ satisfy all required conditions for existence and uniqueness of the solutions for all $t \geq t_0, I_k, J_k \in C(R^n, R^n), C$ is an open set in $PC([-\tau, 0], R^n)$ The time sequence $[t_k]_{k=1}^{+\infty}$ satisfy 0=$t_0 < t_1 < \cdots < t_k < \cdots, \lim_{k\to 0} t_k = \infty$ and $x_t, x_{t^-} \in PC([-\tau, 0], R^n)$ are defined by $x_t(r) = x(t + r)$,for $-\tau \leq r \leq 0$ .We shall assume that f(t,0)= $I_k(t, 0) = 0$ for all t $\in R_+$ and k $\in N$,so that system (i) admits the trivial solution.

Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $\|.\|_\tau = sup_{-\tau \leq r \leq 0}\|\psi(r)\|$.Denote x(t)=x(t,$t_0$, φ),for be the unique solution of (1).Also, assume that $f(t, 0) = 0, I_k(0) = 0$ and $J_k(0) = 0, t \in R_+$ and $k \in Z_+$,so that the system (1) admits zero solution.

In this paper, we make the following assumptions.

H1: For $t \in [t_0 - \tau, t_0]$, the solution $x(t, t_0, \varphi)$ coincides with the function $\varphi(t - t_0)$.

H2: $f(t, \varphi)$ is Lipchitzian in φ in each compact set in $PC([-\tau, 0], R^n)$.

H3: $\varphi + I_k(\varphi(0)) + J_k(\varphi) \epsilon C, k \epsilon N$.

H4: For each $x(r): [t_0 - \tau, \infty] \to R^n$,which is continuous everywhere except at the points t_k at which $x(t_k^+), x(t_k^-)$ exist $x(t_k^+) = x(t_k), f(t, x_t)$ is continuous for almost all $t \in [t_0, \infty)$,at the discontinuous points f is right continuous.

Definition 1: The function $V : R_+ \times R^n \to R_+$ is said to belong to the class v_0 if we have the following.

1)V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$,and for each $x, w \in R^n$,$t \in [t_{k-1}, t_k)$ $k \in N$, $lim_{(t,w)\to(t_k^-,x)} V(t, w) = V(t_k^-, x)$ exists.

2) $V(t, x)$ is locally Lipschitzian in all $x \in R^n$,and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2: Given a function $V : R_+ \times R^n \to R_+$, the upper right-hand derivative of V with respect to system (1) is defined by

$D^+V(t, \vartheta(0)) = \lim_{\theta \to 0^+} sup \frac{1}{\theta} [V(t+k, \vartheta(0) + kf(t, \emptyset)) - V(t, \vartheta(0))]$

for $(t, \emptyset) \in R_+ \times PC([-\tau, 0], R^n)$.

Definition 3: Let $x(t) = x(t, t_0, \varphi)$be the solution of system (1) through (t_0, φ).Then the zero solution of the impulsive differential system (1) is said to be stable if, for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists a $\theta = \theta(\varepsilon, t_0) > 0$ such that $\|\varphi\|_\tau < \theta$ implies that $\|x(t, t_0, \varphi)\| < \varepsilon, t \geq t_0$ and if θ is independent of t_0 then it is uniformly stable.

3. MAIN RESULTS

In the following, we shall establish criteria on impulsive differential equation with any time delay for uniform stability. We have the followings results.

Theorem 1: Let the conditions (H1)-(H4) be satisfied. Assume that there exists a function $V \in \vartheta_0$ and some positive constants $q, w, w_1, w_2 > 0$ and $\rho > 1$,such that

(a) $w_1\|x\|^q \leq V(t,x) \leq w_2\|x\|^q$,for any t $\in [t_0-\tau,\infty)$ and x $\in R^n$.

(b) $D^+V(t,x(t)) \leq wV(t,x(t))$,for all $t \in [t_{k-1}, t_k)$ $k \in N$

Whenever $\delta\, V(t,x(t)) \geq V(t+r,x(t+r))$,where r $\in [-\tau,0]$.

(c) $V(t_k, I_k(x(t_k^-)) + J_k(x(t_k^- - \tau))) \leq \frac{1+b_k}{2}\left[V(t_k^-, x(t_k^-)) + V(t_k^- - \tau, x(t_k^- - \tau))\right]$,where $b_k \geq 0, \sum_{k=1}^{\infty} b_k < \infty$ and $\frac{1+b_k}{2} \leq \frac{1}{\rho}$.

(d) $(t_k - t_{k-1}) \leq u < \frac{\ln\rho}{w}, \forall k \in N$.

Here x(t) is the solution of the system (1).Then for any time delay $\tau \in (0,\infty)$,the zero solution of impulsive differential equation (1) is uniformly stable.

Proof: Let x(t)=x(t, t_0, φ) be any solution of the impulsive system (1).

Now, we will show that

$V(t,x(t)) \leq \rho w_2\|\varphi\|_\tau^q, t \in [t_{k-1}, t_k), k \in N.$ (2)

Firstly, we will prove that

$V(t,x(t)) \leq \rho w_2\|\varphi\|_\tau^q, t \in [t_0, t_1).$ (3)

Let us suppose that there exists $t^* \in [t_0,t_1)$, such that $V(t^*, x(t^*)) > \rho w_2\|\varphi\|_\tau^q$.

Let $t' = \inf\{t \in (t_0, t') | V(t,x(t)) > \rho w_2\|\varphi\|_\tau^q\}$.

Also $V(t_0, x(t_0)) \leq w_2\|\varphi\|_\tau^q < \rho w_2\|\varphi\|_\tau^q$,where $t' > t_0$.

Also we know that $V(t', x(t')) = \rho w_2\|\varphi\|_\tau^q, V(t,x(t)) \leq \rho w_2\|\varphi\|_\tau^q, t \in [t_0-\tau, t']$ (4)

Define $t^\# = \sup\{t \in (t_0, t') | V(t,x(t)) > \rho w_2\|\varphi\|_\tau^q\}$.Here $t^\# < t'$ and

$V(t^\#, x(t^\#)) = w_2\|\varphi\|_\tau^q, V(t,x(t)) > w_2\|\varphi\|_\tau^q, t \in (t^\#, t']$ (5)

By equation (4) and (5), we get

$\rho V(t,x(t)) \geq \rho\, w_2\|\varphi\|_\tau^q \geq V(r,x(r)), t-\tau \leq r \leq t.$

So by the condition (ii), we obtain that

$\rho w_2\|\varphi\|_\tau^q = V(t',x(t')) \leq V(t^\#, x(t^\#))e^{w(t'-t^\#)} \leq w_2\|\varphi\|_\tau^q e^{wu}$

By condition (iv), this is a contradiction. So, it is proved that equation (3) holds. Also

$V(t_1, x(t_1)) = V(t_1, I_k(x(t_k^-)) + J_k(x(t_k^- - \tau)))$

$\leq \frac{(1+b_k)}{2}(V(t_k^-, x(t_k^-)) + V(t_k^- - \tau, x(t_k^- - \tau)))$

$\leq \frac{(1+b_k)}{2}(w(t_1) + w(t_1 - \tau))$

$\leq \frac{(1+b_k)}{2}w(t_1) \leq \frac{1}{\rho}w(t_1)$

It follows from the equation (3) that

$V(t_1, x(t_1)) \leq \frac{1}{\rho}w(t_1)\rho w_2\|\varphi\|_\tau^q \leq w_2\|\varphi\|_\tau^q$

On the basis of above claim we can prove that

$V(t,x(t)) \leq \rho w_2\|\varphi\|_\tau^q, t \in [t_1, t_2)$

In the similar way, in general we can prove that the equation (2) holds.

For any given $\varepsilon > 0$, choose a constant $\delta > 0$ such that $(\frac{\rho w_2}{w_1})^{\frac{1}{q}}\delta < \varepsilon$.When $\|\varphi\|_\tau < \delta$,it means that $\|x(t)\| < \epsilon, t > t_0$.Thus, for any time delay $\tau \epsilon (0, \infty)$,the zero solution of the impulsive differential system (1) is uniformly stable.

Corollary 1.Let the conditions (H1)-(H4) be satisfied. Assume that there exists a function V$\in \vartheta_0$ and some positive constants $q, w, w_1, w_2 > 0$ and $\rho > 1$, such that

(a) $w_1\|x\|^q \leq V(t,x) \leq w_2\|x\|^q$, for any t $\in [t_0 - \tau, \infty)$ and x $\in R^n$.

(b) $D^+V(t,x(t)) \leq wV(t,x(t))$,for all $t \in [t_{k-1}, t_k)$ $k \in N$

Whenever $\delta\, V(t,x(t)) \geq V(t+r, x(t+r))$, where r $\in [-\tau, 0]$.

(c) $V(t_k, I_k(x(t_k^-))) \leq b_k[V(t_k^-, x(t_k^-)]$

,where $b_k \geq 0, \sum_{k=1}^{\infty} b_k < \infty$ $and\ b_k \leq \frac{1}{\rho}$.

(d) $(t_k - t_{k-1}) \leq u < \frac{\ln \rho}{w}, \forall k \in N$.

Then for any time delay $\tau \in (0, \infty)$, the zero solution of impulsive differential equation (1) is uniformly stable.

4. CONCLUSION

In this paper, we studied the concept of uniform stability criteria of system of impulsive functional differential equations in which state variables on impulses are related to time delay. By using Lyapnov functions and Razumikhin technique, we have gotten some results for the uniform stability of impulsive functional differential equations with any time delay.

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