Laguerre function over Symmetric Cone &their recursion relation on certain Hilbert Spaces of Holomorphic function over T(Ω)

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Abstract

In this Paper, we present some basic definitions and facts regarding Laguerre functions over symmetric cones, and their relation to representations of Lie groups and Lie algebras. We are particularly interested in Laguerre functions defined over $\Omega = Sym^+(\mathbb{R}, n)$ for which we derive their recursion relations through the action of the Lie algebra of the group $Sp(2n, \mathbb{R})$ on certain Hilbert spaces of holomorphic functions over $T(\Omega)$.

Introduction

2.1 Laguerre Functions

The 'classical' Laguerre functions are defined through the Laguerre polynomials that can be defined in many ways. One way is by the *Rondriguez formula*:

Definition 2.1.1. The polynomials defined by $L^\nu_m(x) = \dfrac{e^x x^{-\nu}}{m!} \dfrac{d^m}{dx^m} e^{-x} x^{\nu+m}, x \in \mathbb{R}^+$,

$m, \nu \in \mathbb{N}$, are called **Laguerre polynomials**.

In terms of the hypergeometric function ${}_1F_1$, the Laguerre polynomials $L^\nu_m(x)$ can also be defined as follows:

$$L^\nu_m(x) = \frac{\Gamma(\nu+m+1)}{\Gamma(m+1)} {}_1F_1(-m; \nu+1; x),$$

where:

$$\Gamma(z) = \int_{\mathbb{R}^+} e^{-x} x^{z-1} dx \text{ with } \mathrm{Re}(z) > 0, \text{ and } {}_pF_q(a_\nu; \gamma_s; z) = \sum_{k=0}^{\infty} \frac{\prod_{r=1}^{p} (a_\nu)_k z^k}{\prod_{s=1}^{q} (\gamma_s)_k k!}$$

Recall that $(a_\nu)_k = \dfrac{\Gamma(a_\nu + k)}{\Gamma(a_\nu)}$ and $\Gamma(n+1) = n!$. It is clear to see that the set

$$\left\{\sqrt{\frac{\Gamma(m+1)}{\Gamma(\nu+m+1)}} L^\nu_m(x)\right\} \text{ forms an orthonormal basis for } L^2(\mathbb{R}^+, e^{-x} x^\nu dx).$$

One can define now the Laguerre functions as follows:

Definition 2.1.2. The functions defined by $l^{\nu}_m(x) = e^{-x} L^{\nu}_m(2x)$, $x \in \mathbb{R}^+$, $m, \nu \in \mathbb{N}$, are called **Laguerre functions**.

It is not hard to show now that $\{l^{\nu}_m(x)\}$ forms an orthonormal basis for $L^2(\mathbb{R}^+, x^{\nu} dx)$.

It is also known that l^{ν}_m satisfy certain recursion relations (see [3] p. 273), like the following:

$$x \frac{d^2}{dx^2} l^{\nu}_m(x) + (\nu+1)\frac{d}{dx} l^{\nu}_m(x) + (2m+\nu+1-x) l^{\nu}_m(x) = \quad (2.1)$$

$$x \frac{d^2}{dx^2} l^{\nu}_m(x) + (2x+\nu+1)\frac{d}{dx} l^{\nu}_m(x) + (x+\nu+1) l^{\nu}_m(x) = -2(m+\nu) l^{\nu}_{m-1}(x) \quad (2.2)$$

$$x \frac{d^2}{dx^2} l^{\nu}_m(x) - (2x-\nu-1)\frac{d}{dx} l^{\nu}_m(x) + (x-\nu-1) l^{\nu}_m(x) = -2(m+1) l^{\nu}_{m+1}(x) \quad (2.3)$$

The Laplace Transform of $l^{\nu}_m(x)$ is:

$$L_{\nu}(l^{\nu}_m)(z) = \int_{\mathbb{R}^+} e^{-(z|x)} l^{\nu}_m(x)\, d\mu_{\nu}(x) = \frac{\Gamma(\nu+m+1)}{\Gamma(m+1)} (z-1)^m (z+1)^{-(m+\nu+1)}$$

Denote the polynomials on the right-hand side of the equation above by $q^{\nu}_m(z)$. Then, $\{q^{\nu}_m(z)\}$ is an orthogonal basis of the space of $H_{\nu}(\mathrm{H}, x^{\nu-1}dz)$, where $\mathrm{H} = \mathbb{R} + i\mathbb{R}^+$.

Formulation of Problem

Observe that $\mathbb{R}^+$ is a symmetric cone, $\mathbb{R}$ is a Euclidean Jordan algebra, and highest weight representations of $SL_2(\mathbb{R})$ on $H_{\nu}(\mathrm{H})$ are derived through the action of $SL_2(\mathbb{R})$ on the tube domain H (the classical upper half-plane). In their paper, Davidson, Olafsson and Zhang (see [3], also show that one can generate the classical recursion relations (2.1), (2.2) and (2.3), by transferring the representations mentioned above on the space $L^2(\mathbb{R}^+, x^{\nu} dx)$.

One wants to check now whether this can be done for other cones and Euclidean Jordan algebras. That is, given a symmetric cone $\Omega \subset V$, where V is a Jordan algebra, and a connected semisimple Lie group G, we want to build highest weight representations of G on $H_{\nu}(T(\Omega))$.Then, we want to transfer the representations on $L^2(\Omega, d\mu_{\nu})$ to establish recursion relations for the generalized Laguerre functions, for which we give the basic concepts below. The following cases have been settled:

(1) $\Omega = \mathbb{R}^+$, $V = \mathbb{R}$, $G = SL_2(\mathbb{R})$ $(see[3])$

(2) $\Omega = Herm^+(n, n)$, $V = Herm(n, n)$, $G = SU(n, n)$ $(see[4])$

In this chapter, we will present the results for the following case:

$$(3)\ \Omega = Sym^{+}(n,\mathbb{R}),\ V = Sym(n,\mathbb{R}),\ G = Sp(2n,\mathbb{R}) \qquad (\textit{see also}[1]).$$

Note that case (1) was done for $T(\Omega) = V + i\Omega$ (the classical upper half-plane H), and gives the classical recursion relations including (2.1), (2.2) and (2.3) (see [3]). Case (2) was done for $T(\Omega) = \Omega + iV$ (right half-plane) and gives a generalization of the classical relations (see [4]). Case (3) is treated here also for the right half-plane, and gives a generalization of the classical relations as well.

2.1.1 L-invariant Polynomials

Let E_{ii} be the diagonal n x n matrix with 1 in the ii-position and zeros elsewhere. Then $\{E_{11}, \ldots, E_{nn}\}$ is a Jordan frame for $V = Sym(n,\mathbb{R})$. Let $V^{(k)}$ be the +1-eigenspace of the idempotent $E_{11} + \ldots + E_{kk}$ acting on V by multiplication. Each $V^{(k)}$ is a Jordan subalgebra and we have:

$$V^{(1)} \subset V^{(2)} \subset \ldots \subset V^{(n)} = V.$$

If $\det_k$ is the determinant function for $V^{(k)}$ and P_k is orthogonal projection of V onto $V^{(k)}$ then the function $\Delta_k(x) = \det_k P_k(x)$ is the usual kth principal minor for an n x n symmetric matrix; it is homogenous of degree k. In particular $\Delta(x) := \Delta_n(x) = \det(x)$. Let $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$. We say that $m \geq 0$, if each m_i is a nonnegative integer and $m_1 \geq m_2 \geq \ldots \geq m_n \geq 0$. Let $\wedge = \{m \mid m \geq 0\}$.

For each $m \in \mathbb{C}^n$, we define the **generalized power functions** as follows:

$$\Delta_m(x) = \Delta_1^{m_1 - m_2}(x)\,\Delta_2^{m_2 - m_3}(x) \ldots \Delta_n^{m_n}(x).$$

The degree of Δ_m is $|m| := m_1 + \ldots + m_n$. Observe that each generalized power function extends to a holomorphic polynomial on $V_{\mathbb{C}} = Sym(n,\mathbb{C})$ in a unique way.

For each $m \in \wedge$, we define an L-invariant polynomial ψ_m on $J_{\mathbb{C}}$ by:

$$\psi_m(z) = \int_L \Delta_m(lz)\, dl, \quad z \in V_{\mathbb{C}}$$

where L is the group that fixes e in Ω and dl is the normalized Haar measure on L. Notice that for the case of H, i.e. n=1, we have $\psi_m(z) = \psi_m(z) = z^m$, as $L = \{1\}$.

Lemma 2.1.3. If $P(V_{\mathbb{C}})$ is the space of all polynomial functions on $V_{\mathbb{C}}$ and $P(V_{\mathbb{C}})^L$ denotes the space of L-invariant polynomials, then $\{\psi_m\}_{m\geq 0}$ is a basis of $P(V_{\mathbb{C}})^L$. Furthermore, if $P_k(V_{\mathbb{C}})_L$ denotes the space of L-invariant polynomials of degree less than or equal k, then $\{\psi_m\}_{|m|\leq k}$ is a basis of $P_k(J_{\mathbb{C}})^L$.

Proof. See [22], p. 61-90.

The lemma above implies that $\psi_m(e+x)$ is a linear combination of $\psi_n\,|n|\leq|m|$. This allows us to define the generalized binomial coefficients $\binom{m}{n}$ from the equation:

$$\psi_m(e+x)=\sum_{|n|\leq|m|}\binom{m}{n}\psi_n(x).$$

2.1.2 The Generalized Gamma Function

The **generalized Gamma function** is defined as follows:

$$\Gamma_\Omega(m)=\int_\Omega e^{-tr(x)}\Delta_m(x)\,\Delta(x)^{-\frac{d}{r}}dx,$$

where $x\in\Omega$ *and* $m\in\wedge$. The numbers d and r are, respectively, thedimension and the rank of the Jordan algebra $V=Sym(n,\mathbb{R})$. Convergence conditions for the integral above, and other properties of $\Gamma_\Omega(m)$, are given in the following proposition.

Proposition 2.1.4. Let $m=(m_1,m_2,\ldots,m_n)\in\mathbb{C}^n$. Then the following hold:

1. The integral defining $\Gamma_\Omega(m)$ converges if $\Re e(m_j)>\frac{1}{2}(j-1)$, where j=1, …, n.

Furthermore,

$$\Gamma_\Omega(m)=(2\pi)^{\frac{n(n-1)}{4}}\prod_{i=1}^{n}\Gamma\left(m_i-\frac{1}{2}(i-1)\right),$$

Where Γ is the classical Gamma function.

2. Take $e_j=(0,\ldots,0,1,0,\ldots,0)^t$, with 1 in the jth position. Then, $\forall m\in\mathbb{C}^n$, we have:

$$(a)\ \frac{\Gamma_\Omega(m)}{\Gamma_\Omega(m-e_j)} = m_j - 1 - \frac{1}{2}(j-1)$$

$$(b)\ \frac{\Gamma_\Omega(m+e_j)}{\Gamma_\Omega(m)} = m_j - 1 - \frac{1}{2}(j-1)$$

Proof. See Theorem VII.1.1 in [8], p. 123, for (1). Part (2) follows easily from (1).

Let $\lambda \in \mathbb{R}$. We correspond λ to the multi-index (λ, …, λ)00000 and denote the latter by λ as well. The context of use should distinguish the two. Then, we define:

$$(\lambda)_m = \frac{\Gamma_\Omega(m+\lambda)}{\Gamma_\Omega(\lambda)}$$

2.1.4 The Generalized Laguerre Functions

Let ν>0 and m∈Λ. Then, the **generalized Laguerre polynomials** are defined (see [8], p. 343) by:

$$L^\nu_m(x) = (\nu)_m \sum_{|n| \leq |m|} \binom{m}{n} \frac{1}{(\nu)_n} \psi_n(-x), \quad x \in \Omega$$

The **generalized Laguerre functions** are defined in terms of $L^\nu_m(x)$ by:

$$l^\nu_m(x) = e^{-tr(x)} L^\nu_m(2x).$$

Remark 2.1.5. Notice that for $\Omega = \mathbb{R}^+$, i.e. n=1, the generalized Laguerre polynomials and functions defined above are precisely the classical Laguerre polynomials and functions defined on $\mathbb{R}^+$ (see [3]).

Recall that from Prop. 1.1.11 (b) we know that the measure $d_p x = \Delta(x)^{-\frac{p}{2}} dx$, where $p = \frac{2d}{r}$, is an H-invariant measure on Ω. Define now the following measure:

$$d\mu_\nu(x) = \Delta(x)^{\nu - \frac{p}{2}} dx.$$

Theorem 2.1.6. The set $\{l^\nu_m\}_{m \geq 0}$ is an orthogonal basis of $L^2_\nu(\Omega, d\mu_\nu)^L$, the Hilbert space of L-invariant functions in $L^2_\nu(\Omega, d\mu_\nu)$.

Proof. See Theorem 7.8 in [4], p. 191

Finally, observe that by Prop. 1.1.11 (b) it follows that H acts unitarily on $L^2_\nu(\Omega, d\mu_\nu)$. by the formula:

$$\lambda_{\nu}(h) f(x) = \det(h)^{\nu} f(h^{t}.x).$$

2.2 $Sp(2n,\mathbb{R})$ **and Its Lie Algebra** $sp(2n,\mathbb{R})$

In this section, we use a non-standard model of $Sp(2n,\mathbb{R})$ suitable for the action on the right half-plane. This non-standard model is a group G (S) isomorphic to $Sp(2n,\mathbb{R})$. We describe some important subgroups of G(S). Then, we introduce some subalgebras of $g_{\mathbb{C}}$, the complexification of the Lie algebra of G(S).

The group $Sp(2n,\mathbb{R})$ is called the symplectic group and is usually defined as:

$$Sp(2n,\mathbb{R}) = \{ g \in SL(2n,\mathbb{R}) \mid g^{t} J g = J \},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Defined this way, $Sp(2n,\mathbb{R})$ acts on the upper half-plane by linear transformations.

Consider now the map:

$$Sp(2n,\mathbb{R}) \ni \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix} \xmapsto{P^{*}(.)P} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(S) \subset SL(2n,\mathbb{C}),$$

where $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. This map is an isomorphism, in other words $P^{*} Sp(2n,\mathbb{R}) P \cong G(S)$

Hence, the group G(S) is an isomorphic copy of $Sp(2n,\mathbb{R})$ in $SL(2n,\mathbb{C})$, and it can be defined as follows:

$$G(S) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2n,\mathbb{C}) \;\middle|\; \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix} \in Sp(2n,\mathbb{R}) \right\}$$

As in p. 28, we note once again that $G(T(\Omega)) \cong G(S)/\mathbb{Z}$, where $\mathbb{Z}$ denotes the center of G(S), but we will actually use G(S) the for the action on the right-half plane T(Ω). Consequently, by definition of G(S), we have the following relations among A, B, C, and D:

$A^{t}C - C^{t}A = 0$	$AB^{t} - BA^{t} = 0$
$A^{t}D - C^{t}B = I$	$AD^{t} - BC^{t} = I$
$B^{t}D - D^{t}B = 0$	$CD^{t} - DC^{t} = 0$
$B^{t}C - D^{t}A = -I$	$AD^{t} - BC^{t} = -I$

We will use precisely this copy of Sp (2n, $\mathbb{R}$), namely G(S), to realize an action of Sp (2n, $\mathbb{R}$) on the right half-plane. This action of G(S) on T (Ω), where T (Ω)= S= Sym_+ (n, $\mathbb{R}$)+ iSym (n, $\mathbb{R}$), is given b y linear transformations as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} . z = (Az + B)(Cz + D)^{-1}.$$

It is already shown in Example 1.4.3 that the action above is well defined, using the fact that z $\in$ T(Ω) if and only if

$$\frac{z + z^*}{2} > 0$$

Some important subgroups of G(S) are the following:

$$K = Stab(I) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G(S) \mid A \pm B \in U(n) \right\} \cong U(n)$$

$$H = G(\Omega) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A)^{t-1} \end{pmatrix} \in G(S) \mid A \in GL(n, \mathbb{R}) \right\} \cong GL(n, \mathbb{R})$$

and

$$L = K \cap H = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in G(S) \mid A \in SU(n) \right\} \cong SU(n),$$

where K is the stabilizer of the identity (it is maximal compact), H is the group that fixes Ω, and L is the intersection of K and H. We found H by observing that ix + (ix)*=0, ∀ x ∈Ω, as the following proposition suggests:

2.3 Representations of sp (2n, $\mathbb{R}$) $\mathbb{C}$ on H_ν (T(Ω))

In this section we want to build representations of G(S) and its Lie Algebra g on H_ν(S), the Hilbert space of holomorphic functions on S. Since here T (Ω)= S= Sym^+ (n,)$\mathbb{R}$ iSym (n,), that$\mathbb{R}$means $d = \frac{n(n+1)}{2}$ and r=n.

2.3.1 The Hilbert Space H_ν (T(Ω))

Consider the following Hilbert space of holomorphic functions:

$$H_\nu(T(\Omega)) = \{ F \mid F : T(\Omega) \rightarrow \mathbb{C}, \| F \|_2 < \infty \},\ \nu \in \mathbb{R},$$

where

$$\| F \|^2 = \alpha_\nu \int_{T(\Omega)} | F(x + iy) |^2 \Delta(x)^{\nu - (n+1)} dxdy,$$

with Δ(x)= det (x) and $\alpha_\nu = \dfrac{2^{n\nu}}{(4\pi)^{\frac{n(n+1)}{2}} \Gamma_\Omega(\nu - \frac{n+1}{2})}$. Recall that in general:

$$\Gamma_\Omega(m) = \int_\Omega e^{-tr(x)} \Delta_m(x)^{-\frac{d}{r}} dx$$

and

$\Delta_m(x) = \Delta_1^{m1-m2}(x)\Delta_2^{m2-m3}(x)\ldots\Delta_n^{mn}(x)$.

It is clear that the norm came from the inner product on H_ν (T (Ω)), which is defined by :

$$(F \mid G) = \alpha_\nu \int_{T(\Omega)} F(x+iy)\overline{G(x+iy)}\Delta(x)^{\nu-(n+1)} dxdy.$$

Finally, notice that H_ν (T(Ω)) is a **reproducing kernel Hilbert space** (see [4] and [20] for more details). This means that point evaluation $E_z : H_\nu((T(\Omega)) \to \mathbb{C}$ given by E_z (F)=F(z) is continuous, $\forall z \in T(\Omega)$. This implies the existence of a kernel function $K_z \in H_\nu$ (T(Ω), such that F (z)= (F|K_z) for all F∈H_ν (T(Ω)) and z∈T(Ω). Set K (z, w)= K_w (z). Then K(z, w) is holomorphic in the first variable and antiholomorphic in the second variable. The function K(z,w) is called the reproducing kernel for H_ν (T(Ω)). We note that the Hilbert space is completely determined by the function K (z, w). In particular, we have the following theorem:

Theorem 2.3.1. Suppose that ν≥ n+1. Then for the Hilbert space H_ν (T(Ω)) we have:

1. The reproducing kernel of H_ν (T(Ω)) is given by $K_\nu(z, w) = \Gamma_\Omega(\nu)\Delta(z+\bar{w})^{-\nu}$
2. The functions $q_m^\nu(z) := \Delta(z+e)^{-\nu}\psi_m\left(\frac{z-e}{z+e}\right)$, $m \in \wedge$, form an orthogonal basis of $H_\nu(T(\Omega))^L$, the space of L-invariant functions in H_ν (T(Ω)).

Proof. See Theorem 2.9 in [4], and Prop's XIII. 1.2 and XIII. 1.3 in [8], p. 261, for the proofs.

Remark 2.3.2. We close this discussion with the following remarks:

1. $F \in H_\nu(T\Omega)^L$ if $\pi_\nu(l)F(z) = F(z), \forall l \in L$, where π_ν is a representation of L.
2. $H_\nu(T(\Omega))^o := \{\Sigma c_j K w_j \mid c_j \in \mathbb{C}, w_j \in T(\Omega)\}$, the space of finite linear combinations, is dense in H_ν (T(Ω)).
3. The inner product in H_ν (T(Ω))^o is given by:

$$(\Sigma_j c_j K_{w_j} \mid \Sigma_k d_k K_{z_k}) = \Sigma_{j,k} c_j \overline{d_k} K(z_k, w_j).$$

2.3.2 The Action of sp(2n,)ℝ ℂ on Hν (T (Ω))

The representation of G(S) on Hν (S) is given by a multiplier representation as follows:

$$\pi_\nu(g)F(z) = J(g^{-1}, z)^{\frac{\nu}{p}} F(g^{-1}.z),$$

Where J (g,z) is complex Jacobian of the action of G(S) on S, i.e. J (g,z) = det D (g.z), and $p = \frac{2d}{r}$. In our case, J(g,z)= det (Cz+D) $^{-(n+1)}$, whenever $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The Lie algebra representation is given, by differentiation, as follows:

$$\pi_\nu(X)F(z) = \frac{d}{dt}\pi_\nu(\exp(tX))F(z)|_{t=0}$$
$$= \frac{d}{dt} J(\exp(-tX), z)^{\frac{\nu}{p}} F(\exp(-tX).z)|_{t=0}$$

Proposition 2.3.3. For each piece of the Lie algebra $g_{\mathbb{C}}$, we have:

1. $\pi_\nu(X)F(z) = -\nu\, tr(az + a)F(z) + D_{\upsilon(a,z)}F(z), X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in p^+$
2. $\pi_\nu(X)F(z) = -\nu\, tr(-az + a)F(z) + D_{\upsilon(-a,-z)}F(z), X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in p^-$
3. $\pi_\nu(X)F(z) = \nu\, tr(bz)F(z) + D_{w(a,b,z)}F(z), X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \wp_{\mathbb{C}}$

where υ(a,z)=-az-a-zaz-za and w (a,b,z)= -az-b+zbz+za.

Proof. We prove the Proposition case by case:

Case (1): Let X ∈ p^+. Then, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$. Now, as $X^n = 0$ for n≥ 2, we have:

$$\exp(-tX) = \begin{pmatrix} 1-ta & -ta \\ ta & 1+ta \end{pmatrix}$$

As J (g,z)= det (Cz+D)$^{-(n+1)}$, whenever $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we also have:

$$J(\exp(-tX), z) = \det(taz + 1 + ta)^{-(n+1)}.$$

Hence, using also the fact that:

$$\det'(1+tu) = tr(u)$$

we have:

$$\pi_\nu(X)F(z) = \frac{d}{dt} J(\exp(-tX), z)^{\frac{\nu}{n+1}} F(\exp(-tX).z)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\det(taz+1+ta)^{-\nu} F\left(\frac{(1-ta)z+(-ta)}{taz+1+ta}\right)\right]_{t=0}$$

$$= -\nu \det(taz+1+ta)^{-\nu-1} \det'(1+t(az+a))F\left(\frac{(1-ta)z-ta}{taz+1+ta}\right)\Big|_{t=0}$$

$$+ \det(taz+1+ta)^{-\nu} F'\left(\frac{(1-ta)z-ta}{taz+1+ta}\right)\Big|_{t=0}$$

$$= -\nu \det(1)^{-\nu-1} tr(az+a)F(z) + \det(1)^{-\nu} F'(z)\left(\frac{(1-ta)z-ta}{taz+1+ta}\right)'\Big|_{t=0}$$

$$= -\nu\, tr(az+a)F(z) + F'(z)(-az-a-zaz-za)$$

$$= -\nu\, tr(az+a)F(z) + D_{-az-a-zaz-za}F(z)$$

$$= -\nu\, tr(az+a)F(z) + D_{\upsilon(a,z)}F(z).$$

Case (2): Let X ∈ p^-. Then, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}$. Again, since $X^n=0$ for n≥2, we have:

$$\exp(-tX) = \begin{pmatrix} 1-ta & ta \\ -ta & 1+ta \end{pmatrix}$$

Since J(g, z)= det (Cz + D) $^{-(n+1)}$, whenever $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have:

J (exp (-tX), z)= det (-taz + 1 + ta) $^{-(n+1)}$.

Therefore

$$\pi_{\nu}(X)F(z)=\frac{d}{dt}J(\exp(-tX),z)^{\frac{\nu}{n+1}}F(\exp(-tX).z)|_{t=0}$$

$$=\frac{d}{dt}\left[\det(-taz+1+ta)^{-\nu}F\left(\frac{(1-ta)z+ta}{-taz+1+ta}\right)\right]_{t=0}$$

$$=-\nu\det(-taz+1+ta)^{-\nu-1}\det'(1+t(-az+a))F\left(\frac{(1-ta)z+ta)}{-taz+1+ta}\right)\Big|_{t=0}$$

$$+\det(-taz+1+ta)^{-\nu}F'\left(\frac{(1-ta)z+ta}{-taz+1+ta}\right)\Big|_{t=0}$$

$$=-\nu\det(1)^{-\nu-1}tr(-az+a)F(z)$$

$$+\det(1)^{-\nu}F'(z)\left(\frac{(1-ta)z-ta}{taz+1+ta}\right)'\Big|_{t=0}$$

$$=-\nu tr(-az+a)F(z)+F'(z)(-az-a+zaz-za)$$

$$=-\nu tr(-az+a)F(z)+D_{-az+a+zaz-za}F(z)$$

$$=-\nu tr(-az+a)F(z)+D_{\upsilon(-a,-z)}F(z).$$

Case (3); Let $X\in\wp_{\mathbb{C}}$ Then, $X=\begin{pmatrix}a & b\\ b & a\end{pmatrix}$, and for exp (-tX) we have:

$$\exp(-tX)=\begin{pmatrix}1-ta+\frac{t^2}{2!}(a^2+b^2)-\dots & -tb+\frac{t^2}{2!}(ab+ba)-\dots\\ -tb+\frac{t^2}{2!}(ab+ba)-\dots & 1-ta+\frac{t^2}{2!}(a^2+b^2)-\dots\end{pmatrix}$$

Similarly, J (g,z)= det (Cz + D)$^{-(n+1)}$, whenever $g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$.Therefore,

$$J(\exp(-tX),z)=\det\left(\left(-tb+\frac{t^2}{2!}(ab+ba)-\dots\right)z+1-ta+\frac{t^2}{2!}(a^2+b^2)-\dots\right)^{-(n+1)}$$

Hence,

$$\pi\nu(X)F(z)=\frac{d}{dt}J(\exp(-tX),z)^{\frac{\nu}{n+1}}F(\exp(-tX).z)\Big|_{t=0}$$

$$=\frac{d}{dt}\left[\det\left(\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)^{-\nu}F\left(\frac{\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)z+\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)}{\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)}\right)\right]_{t=0}$$

$$=-\nu\det\left(\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)^{-\nu-1}$$

$$\det'(1+t((-b+t(ab+ba)-...)z-a+t(a^2+b^2)-...))$$

$$F\left(\frac{\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)z+\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)}{\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)}\right)\Bigg|_{t=0}$$

$$+\det\left(\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)^{-\nu}$$

$$F'\left(\frac{\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)z+\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)}{\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)}\right)\Bigg|_{t=0}$$

$$=-\nu\det(1)^{-\nu-1}tr(-bz-a)F(z)+\det(1)^{-\nu}F'(z)$$

$$\left(\frac{\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)z+\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)}{\left(-tb+\frac{t^2}{2!}(ab+ba)-...\right)z+\left(1-ta+\frac{t^2}{2!}(a^2+b^2)-...\right)}\right)\Bigg|_{t=0}$$

$$=-\nu\,tr(-bz-a)F(z)+F'(z)(-az-b+zbz+za)$$

$$=\nu\,tr(bz)F(z)+D_{-az+a+zaz-za}F(z),\ \text{since}\quad tr(a)=0$$

$$=\nu\,tr(bz)F(z)+D_{w(a,b,z)}F(z)$$

2.3.3 Highest Weight Representations

In this subsection, we introduce the basic concepts on highest weight representations. The representation π_ν we constructed in the previous subsection is a highest weight representation, and that is important for the recursion relations for Laguerre functions.

Suppose that G is a Hermitian group. That means G is simple and the maximal compact subgroup K has dim Z(K)=1. The Hermitian groups have been classified in terms of their Lie algebras, where the latter are $su(p,q)$, $sp(n,)$, $so*(2n)$, $so(2,n)$ and two exceptional Lie algebras. The fact that Z (K)=1 implies that G/K is isomorphic to a bounded symmetric complex domain. It also implies that $g_{\mathbb{C}}$ has a decomposition of the form $g_{\mathbb{C}} = p_{+} \oplus \wp_{\mathbb{C}} \oplus p_{-}$, where $\wp_{\mathbb{C}}$ is the Lie algebra of $K_{\mathbb{C}}$ The subspaces p_{+}, $\wp^{\mathbb{C}}$ and p_{-} are, respectively, the -2, 0, 2-eigenspaces of ad(z), for some $z \in Z(\wp_{\mathbb{C}})$.

Lemma 2.3.4. We have the following inclusions for the spaces p_{+}, $\wp^{\mathbb{C}}$ and p_{-} :

$$(a)\ [p^{+}, p^{-}] \subset \wp^{\mathbb{C}}$$
$$(b)\ [\wp, p^{\pm}] \subset p^{\pm}$$

Proof. (a) Suppose that $z \in Z(\wp_{\mathbb{C}})$ and let $X \in p_{+}$, $Y \in p_{-}$ and $Z \in \wp_{\mathbb{C}}$ Then we have

$$ad(z)[X,Y] = [ad(z)X,Y] + [X, ad(z)Y] = -2[X,Y] + 2[X,Y] = 0,$$

which implies that $[X,Y] \in \wp_{\mathbb{C}}$.

(b) Similarly, ad $(z)[Z,X] = [ad(z)Z, X] + [Z, ad(z)X] = -2[Z,X]$, which implies that $[Z,X] \in p_{+}$. Similar calculations give that $[Z,Y] \in p_{-}$.

Suppose π that is an irreducible representation of G on Hilbert Space H. We say π is a highest weight representation if there is a nonzero vector $\upsilon \in \mathrm{H}$ such that

$$\pi(X)\upsilon = 0,\ \forall X \in p^{+}.$$

Let $\mathrm{H}_{o} = \{ \upsilon \in \mathrm{H} \mid \pi(X)\upsilon = 0,\ \forall X \in p^{+} \}$. Then, we have the following important theorem.

Theorem 2.3.5. Suppose π is an irreducible unitary highest weight representation of G on H and H_o is defined as above. Then $(\pi|_K, H_o)$ is irreducible. Furthermore, there is a scalar λ such that:

$$\pi(Z)\upsilon = \lambda\upsilon,\ \forall \upsilon \in H_o$$

If $\mathrm{H}_{n} = \{ \upsilon \in \mathrm{H} \mid \pi(z)\upsilon \}$, then $\mathrm{H} = \oplus_{n \geq 0} \mathrm{H}_{n}$. Furthermore, we have:

$$\pi(Z): H_n \to H_n, \quad Z \in \wp^{\mathbb{C}}$$
$$\pi(X): H_n \to H_{n-1}, X \in p^+$$
$$\pi(Y): H_n \to H_{n+1}, \quad Y \in p^-,$$

where, in the case n=0, H_{-1} is understood to be the {0} space.

Proof. By Lemma 2.3.4, H_o is an invariant K-space. Suppose V_o is a nonzero invariant subspace of H_o and W_o is its orthogonal complement in H_o. Define V_n inductively as follows:

$$V_n = span\left\{\pi(Y)\upsilon \mid Y \in p^-, \upsilon \in V_{n-1}\right\}.$$

Let $V = \oplus V_n$. Define W_n in the same way as V_n and let $W = \oplus W_n$. Then, by Lemma 2.3.4, V and W are invariant $g^{\mathbb{C}}$ subspaces of H. Since π is unitary V and W are orthogonal. However, since π is irreducible and V is nonzero, it follows that V=H and hence W=0. This implies W_o=0 and thus π/K is irreducible. Since (π(Z) commutes with π (K) Schur's lemma implies that π(z)=λI on H_o for some scalar λ. Since V_o= H_o, induction, Lemma 2.3.4, and irreducibility of π implies that V_n=H_n. The remaining claims follow from Lemma 2.3.4.

Remark 2.3.6. The operators π(X), X∈p^+, are called annihilation operators because, for υ in the algebraic direct sum ⊕ H_n, sufficiently many applications of π(X) annihilates υ. The operators π (Y), Y∈p^-, are called creation operators.

Remark 2.3.7. For $H_\nu(T(\Omega))$, a straightforward calculation gives:

$$\pi_\nu(X)\, q^\nu{}_0 = 0, \forall X \in p^+ .$$

Indeed, observe that $q^\nu{}_0 = \Delta(z+e)^{-\nu}$ from Theorem 2.3.1. Then use Prop. 2.3.3 (1) and (*) from the proof of Lemma 2.4.2. Finally, we also have that $H_\nu(T(\Omega))_o \ni \mathbb{C}\, q^\nu{}_0$, which says that π_ν is an irreducible unitary highest weight representation of G.

2.4 Representations of $sp(2n,\mathbb{R})_{\mathbb{C}}$ $on\ L^2(\Omega, d\mu_\nu)$

In the previous section we have seen representations of g on $H_\nu(T(\Omega))$. In this section we want to build representations of g on $L^2(\Omega, d\mu_\nu)$. We will actually transfer the previous representations that are on $H_\nu(T(\Omega))$ $to\ L^2(\Omega, d\mu_\nu)$ with the help of the Laplace transform.

2.4.1 The Laplace Transform

Consider the space $L^2(\Omega, d\mu_\nu)$, where $d\mu_\nu(x) = \Delta(x)^{\nu-\frac{d}{r}}$. The Laplace transform is defined by:

we have the

$$\pi_\nu(X)L_\nu(f)(z) = L_\nu(tr(bx)f)(z) + L_\nu(tr((ax - xa)\nabla)f)(z) - \nu L_\nu(tr(b\nabla)f)(z) - L_\nu(tr(b\nabla x\nabla)f)(z).$$

Finally taking L^*_ν in both sides, and considering (2.4), we get:

$$\lambda_\nu(X)f(x) = tr(\nu bx + (ax - xa - \nu b)\nabla - b\nabla x\nabla)\ f(x),\ X \in \wp_{\mathbb{C}}.$$

2.5 Recursion Relations for l^ν_m

Recall the functions defined in Theorem 2.3.1 given by $q^\nu_m(z) = \Delta(z+e)^{-\nu}\psi_m\left(\frac{z-e}{z+e}\right)$. As shown, these functions form an orthogonal basis of $H_\nu(T(\Omega))^L$. Recall also, that $\wp_{\mathbb{C}}$ and $h_{\mathbb{C}}$ denote the Lie algebras of $K_{\mathbb{C}}$ and $H_{\mathbb{C}}$ respectively.

Proposition 2.5.1. The Laguerre functions l^ν_m relate with $q^\nu_m(z)$ as follows:

$$L(l^\nu_m)(z) = \Gamma_\Omega(m+n)\, q^\nu_m(z)$$

Proof Sec [4] p. 187-191, and [8] p. 344. For the classical case, see [3] p. 271-273.

From Proposition 2.3.3 and Theorem 2.4.5, we can actually obtain the corresponding relations for q^ν_m and l^ν_m respectively. Concerning q^ν_m we know (by Lemma 5.5 in [4], p. 182) that, for $\xi \in Z(\wp_{\mathbb{C}})$ and $Z_0 \in Z(h_{\mathbb{C}})$ (can take $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), where $Z(\wp_{\mathbb{C}})$ and $Z(h_{\mathbb{C}})$ denote the centers of $\wp_{\mathbb{C}}$ and $h_{\mathbb{C}}$ respectively, we have:

$$\pi_\nu(\xi)\, q^\nu_m(z) = (r\nu + 2|m|)\, q^\nu_m(z) \qquad (2.5)$$

and

$$\pi_\nu(-2Z_0)\, q^\nu_m(z) = \sum_{j=1}^{r}\binom{m}{m-e_j} q^\nu_{m-e_j} - \sum_{j=1}^{r}\left(\nu + n_j - \frac{s}{2}(j-1)\right) c_m(j)\, q^\nu_{m+e_j}(z) \qquad (2.6)$$

where : $e_j = (0,........0,1,0.......,0)^t$, with 1 in the jth position

and

$$c_m(j) = \prod_{j\neq k} \frac{n_j - n_k - \frac{s}{2}(j+1-k)}{n_j - n_k - \frac{s}{2}(j-k)}$$

Theorem 2.5.2. The Laguerre functions satisfy the following differential recursion relations:

1. $(r\nu + 2|m|) l^{\nu}_{m} = tr(x - \nu\nabla - \nabla x\nabla) l^{\nu}_{m}$
2. $\sum_{j=1}^{r} c_m(j) l^{\nu}_{m+e_j} = tr(\nu e - x + (2x - \nu)\nabla - \nabla x\nabla) l^{\nu}_{m}$
3. $-\sum_{j=1}^{r} \binom{m}{m-e_j}\left(m_j - 1 + \nu - (j-1)\frac{s}{2}\right) l^{\nu}_{m-e_j} = tr(\nu e + x + (2x + \nu)\nabla + \nabla x\nabla) l^{\nu}_{m}$

Proof. Transferring representation (2.5) above onto $L^2(\Omega, d\mu_\nu)^L$, we get:

$$\lambda_\nu(\xi) l^{\nu}_{m}(x) = (r\nu + 2|m|) l^{\nu}_{m}(x). \qquad (2.7)$$

Now, combining Proposition 2.4.5 (3) (for $f = l^{\nu}_{m}$), $X = \xi$, i.e. a=0, b=1) and (2.7), we get the first recursion relation for l^{ν}_{m} :

$$(r\nu + 2|m|) l^{\nu}_{m} = tr(x - \nu\nabla - \nabla x\nabla) l^{\nu}_{m} \qquad (2.8)$$

Let $L^2_k(\Omega, d\mu_\nu) = \{ f \in L^2(\Omega, d\mu_\nu) \mid \lambda_\nu(\xi) f = (r\nu + 2k) f \}$. Then as λ_ν are highest weight representations, we have:

$$L(\Omega, d\mu_\nu) = \oplus L^2_k(\Omega, d\mu_\nu),$$

where $L^2_k(\Omega, d\mu_\nu) \neq \{0\}$ if $k \geq 0$. Observe also that $l^{\nu}_{m} \in L^2_{|m|}(\Omega, d\mu_\nu)$, by (2.7).

Now, for $X \in p^+$, we have:

$$\begin{aligned}
\lambda_\nu(\xi)\lambda_\nu(X) f &= \lambda_\nu(X)\lambda_\nu(\xi) f + \lambda_\nu([X, \xi]) f \\
&= \lambda_\nu(X)(r\nu + 2k) f + \lambda_\nu(ad(\xi) X) f \\
&= (r\nu + 2k)\lambda_\nu(X) f + \lambda_\nu(2X) f \\
&= (r\nu + 2(k+1))\lambda_\nu(X) f.
\end{aligned}$$

That is, $\lambda_\nu(X) f \in L^2_{k+1}(\Omega, d\mu_\nu), X \in p^+$. Similarly, $\lambda_\nu(X) f \in L^2_{k-1}(\Omega, d\mu_\nu), X \in p^-$. This says, that $\lambda_\nu(X)$, for X in p^+ and p^- respectively act as k+1 and k-1 projections. Let $X^+ = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in p^+, X^- = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in p^-$ (take a=1). Then,

$$Z_0 = \frac{1}{2}(X^+ + X^-) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $Z_0 \in Z(h_{\mathbb{C}})$, where $Z(h_{\mathbb{C}})$ denotes the center of $h_{\mathbb{C}}$. Transferring representation (2.6) onto $L^2(\Omega, d\mu_\nu)^L$ we get:

$$-\lambda_\nu(-2Z_0)l^\nu_m(x)=\sum_{j=1}^{r}\binom{m}{m-e_j}\left(m_j-1+\nu-(j-1)\frac{s}{2}\right)_\nu l^\nu_{m-e_j}(x)-\sum_{j=1}^{r}c_m(j)l^\nu_{m+e_j}(x). \quad (2.9)$$

So, for each one of the projections, (2.9) gives:

$$\lambda_\nu(X^-)l^\nu_m(x)=-\sum_{j=1}^{r}\binom{m}{m-e_j}\left(m_j-1+\nu-(j-1)\frac{s}{2}\right)_\nu l^\nu_{m-e_j}(x)-\sum_{j=1}^{r}c_m(j)l^\nu_{m+e_j}(x). \quad (2.10)$$

and

$$\lambda_\nu(X^+)l^\nu_m(x)=\sum_{j=1}^{r}c_m(j)l^\nu_{m+e_j}(x). \quad (2.11)$$

Finally, combining Proposition 2.4.5 (2), (1) (for $f=l^\nu_m$, $X=X^-$ and $X=X^+$, a=1) and (2.10), (2.11) respectively, one obtains the remaining recursion relations for l^ν_m:

$$-\sum_{j=1}^{r}\binom{m}{m-e_j}\left(m_j-1+\nu-(j-1)\frac{s}{2}\right)_\nu l^\nu_{m-e_j}=tr\,(\nu-x+(2x-\nu)\nabla-\nabla x\nabla)l^\nu_m \quad (2.12)$$

and

$$\sum_{j=1}^{r}c_m(j)l^\nu_{m+e_j}=tr\,(\nu+x+(2x+\nu)\nabla+\nabla x\nabla)l^\nu_m \quad (2.13)$$

Notice that when one restricts down to $\mathbb{R}$, equations (2.8), (2.12) and (2.13) correspond to the classical relations (2.1), (2.2) and (2.3) respectively.

Remark 2.5.3. We conclude with the following remarks:

(a) Note that $X_0, X^+, X^- \in g^L_{\mathbb{C}}$, where $g^L_{\mathbb{C}}=\{X\in g_{\mathbb{C}} \mid Ad(l)X=X, \forall l\in L\}$.

(b) Starting from the unit disc D, one could obtain the polynomials $q^\nu_m(z)$ as follows: Consider the functions $\psi_m(z)=\int_L \Delta_m(lz)\,dl$. Then, $\psi_m(z)\in H_\nu(D)$, where $H_\nu(D)=\{F\in \mathcal{O}(D) \mid \|F\|^2<\infty\}$, and $\|F\|^2 = \beta_\nu\int_D |F(z)|^2 dm(z)$, $\mathcal{O}(D)$ denotes the space of holomorphic functions on the disc D, and $\beta_\nu=\dfrac{\Gamma_\Omega(\nu)}{\pi^d\,\Gamma_\Omega(\nu-\frac{d}{r})}$.

Note that ψ_m are L-invariant, and furthermore that $H_\nu(D)^L \cong \oplus_{m\in\wedge}\mathbb{C}\psi_m$. That is, the ψ_m's span the 1-dimensional eigenspaces of the highest weight representation space. It is known that $D \cong T(\Omega)$ through the Cayley transform $c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, so we have $\pi_\nu(c) H_\nu(D) = H_\nu(T(\Omega))$.

Hence, $\pi_\nu(c)\psi_m(z) = \Delta(z+e)^{-\nu}\psi_m\left(\frac{z-e}{z+e}\right) \in H_\nu(T(\Omega))$. Denote the right-hand side of the above equation by $q^\nu_m(z)$.

Then, $\{q^\nu_m(z)\}$ form an orthogonal basis for $H_\nu(T(\Omega))^L$.

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