Contributions to Fixed Point Theory in Cone Metric Spaces and Cone Metric Spaces with *w*-Distance

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Abstract:

Fixed point theorems provide conditions under which maps (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis, topology, and geometry. Over the last 100 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory and physics. Fixed point theory plays an important role in functional analysis, approximation theory, differential equations and applications such as boundary value problems etc.

Fixed point theory broadly speaking demonstrates the existence, uniqueness and construction of fixed points of a function or a family of functions under diverse assumptions about the structure of the domain X (such as X may be a metric space or normed linear space or a topological space) of the concerned functions. The methods of the theory vary over almost all mathematical techniques. There are many works entirely devoted to fixed point theory such as Brouwer L.E.J, Smart D.R, Vanderwalt T

Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces. In addition, he contributed to measure theory, integration, the theory of sets and orthogonal series. In his dissertation, written in 1920, he defined axiomatically what today is referred to as Banach space.

One extension of metric spaces is the so called cone metric space. In cone metric spaces, the metric is no longer a positive number but a vector, in general an element of a Banach space equipped with a cone. In this paper , we extend results of Sami Ullah Khan and Arjamand Bano[8] and prove some common fixed point theorem for a pair of weakly compatible mappings in cone metric spaces using ω – distance on X without using normality in cone metric space.

1. Introduction:

One extension of metric spaces is the so called cone metric space. In cone metric spaces, the metric is no longer a positive number but a vector, in general an element of a Banach space equipped with a cone.

In 1906, the French mathematician Maurice Frechet introduced the concept of metric spaces, although the name "metric" is due to Hausdorff . In 1934, Duro Kurepa, proposed metric spaces in which an ordered vector space is used as the co-domain of a metric instead of the set of real numbers. In literature Metric Spaces with Vector Valued Metrics are known under various names such as Pseudo Metric Spaces, K-Metric Spaces, Generalized Metric Spaces, Cone-Valued Metric Spaces, Cone Metric Spaces, Abstract Metric Spaces and Vector Valued Metric Spaces. Fixed point theory in K-metric spaces was developed by Perov in 1964 .

In 2007, Huang & Xian [2] introduced the notion of a cone metric space and established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone P as follows:

Definition 1.1: (Huang & Xian [2]) Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{ 0 \}$

(ii) $ax + by \in P \ \forall \ x, y \in P$ and non-negative real numbers a and b.

(iii) $P \cap (-P) = \{0\}$.

Definition 1.2 : (L. G. Huang, Z. Xian ,[2])

We define a partial ordering $\leq$ on E with respect to P and $P \subset E$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x << y$ if $y - x \in \text{int } P$, int P denotes the interior of P. We denote by $\| . \|$ the norm on E. The cone P is called normal if there is a number K > 0 such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\| x\| \leq K \| y\| \qquad \ldots 1.2.1$$

The least positive number K satisfying (1.2.1) is called the normal constant of P.

Definition 1.3: (L. G. Huang, Z. Xian ,[2])

A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{ x_n \}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in$ E ,then there is $x \in$ E such that $\lim_{n\to\infty} \|x_n - x\| = 0$

Definition 1.4: (L. G. Huang, Z. Xian ,[2])

E is a real Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P. Let X be a non-empty set and $d: X \times X \to P$ a mapping such that.

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ (non - negativity)

(d_2) $d(x, y) = 0$ if and only if $x = y$.

(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)

(d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality)

Then d is called a cone-metric on X and (X, d) is called a cone metric space.

Example 1.5 : (L. G. Huang, Z. Xian ,[2])

Let $E = R^2, P = \{(x, y) \in E / x, y \geq 0\}, X = R$ and $d : X \times X \to P$ defined by

$d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant.

Then (X , d) be a cone metric space.

Definition 1.6: (L. G. Huang, Z. Xian ,[2])

Let (X, d) be a cone metric space, $x \in X$ and $\{ x_n \}_{n \geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$.

We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(ii) $\{x_n\}_{n\geq 1}$ is said to be a Cauchy Sequence if for every c ∈ E with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(iii) (X, d) is called a complete cone metric space if every Cauchy Sequence in X is convergent.

L. G. Huang, Z. Xian [2] proved some fixed point theorems of contractive mappings, which generalize the existing results in metric spaces such as Banach, Kannan etc.

Theorem 1.7 : (L. G. Huang, Z. Xian [2],Theorem 1)

Let (X, d) be a complete cone metric space , P be a normal cone with normal constant K . Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx,Ty) \leq k\, d(x , y)$, for all $x , y \in X$, where $k \in [0,1)$ is a constant .

Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{ T^n x \}_{n\geq 1}$ converges to the fixed point.

Definition 1.8 :(L. G. Huang, Z. Xian [2],Definition 5)

Let (X,d) be a cone metric space . If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X, then X is called a sequentially compact cone metric space.

Theorem 1.9: (L. G. Huang, Z. Xian [2], Theorem 2)

Let (X,d) be a sequentially compact cone metric space , P be a regular cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx,Ty) < d(x,y)$, for all $x , y \in X$, $x \neq y$. Then T has a unique fixed point in X.

Theorem 1.10 :(L. G. Huang, Z. Xian [2], Theorem 3)

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx,Ty) \leq \lambda (d(Tx,x) + d(Ty,y))$,for all $x,y \in X$, where $\lambda \in [0,½)$ is a constant . Then T has a unique fixed point in X.

For each $x \in X$, the iterative sequence $\{ T^n x \}_{n\geq 1}$ converges to the fixed point.

Theorem 1.11 : (L. G. Huang, Z. Xian [2] , Theorem 4)

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K . Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx,Ty) \leq \lambda\,(d(Tx,y) + d(x,Ty))$ for all $x,y \in X$, where $\lambda \in [0,½)$ is a constant . Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{ T^n x \}_{n\geq 1}$ converges to the fixed point.

In 2008, Rezapour and Hamlbarani [7] , proved that there are no normal cones with normal constant M < 1. Further, in [7] it is shown that for k > 1 there are cones with normal constant M > k. An example of a non normal cone is given in [7]. Further, Rezapour and Hamlbarani [7] obtained generalizations of the results of L. G. Huang, Z. Xian [2] (Theorems 1.19,1.21 and 1.22) by removing the assumption of normality of the cone.

Lemma 1.12 : (Rezapour and Hamlbarani [7] , Lemma 1.1)

Every regular cone is normal.

Definition 1.13: (H. Lakzian and F. Arabyani [6])

Let X be a cone metric space with metric d. Then a mapping $\omega : X\times X \to E$ is called

ω - distance on X if the following conditions are satisfied

i) $0 \leq \omega(x, y)$ for all $x, y \in X$;

ii) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ for all $x, y , z \in X$;

iii) if $x_n \to x$ then $\omega(y,x_n) \to \omega(y,x)$ and $\omega(x_n, y) \to \omega(x,y)$

iv) for any $0 \ll \alpha$, there exists $0 \ll \beta$ such that $\omega(z, x) \ll \beta$ and $\omega(z, y) \ll \beta$ imply

$d(x, y) \ll \alpha$ for all $\alpha, \beta \in E$ … (1.13.1)

Definition 1.14: (H. Lakzian and F. Arabyani [6])

Let X be a cone metric space with metric d, let ω be a ω – distance on X, $x \in X$ and

$\{ x_n \}$a sequence in X. Then $\{x_n\}$ is called a ω- Cauchy sequence whenever for every

$\alpha \in E$, $0 \ll \alpha$, there is a positive integer N such that, for all $m,n \geq N$, $\omega(x_m,x_n) \ll \alpha$.

A sequence $\{x_n\}$ in X is called ω – convergent to a point $x \in X$ whenever for every

$\alpha \in E$, $0 \ll \alpha$, there is a positive integer N such that for all $n \geq N$, $\omega(x,x_n) \ll \alpha$.

(X,d) is a complete cone metric space with ω- distance if every Cauchy Sequence is ω – Convergent.

Definition 1.15: (G. Jungck.and B.E. Rhoades [4])

Let S and T be self mappings of a set X. If $u = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T and u is called a point of coincidence of S and T.

Definition 1.16: (G. Jungck.and B.E. Rhoades [4])

Two self mappings S and T of a set X are said to be weakly compatible if they commute at their coincidence point. i.e; if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Proposition 1.17: (M.Abbas and G. Jungck [1])

Let S and T be weakly compatible self mappings of a set X. If S and T have a unique point of coincidence , i.e; $u = Sx = Tx$, then u is the unique common fixed point of S and T.

Property 1.18: Let (X,d) be a cone metric space. If $\{x_n\}$, $\{y_n\}$ are sequences in X and $x_n \to x$ and $y_n \to y$ then $d(x_n,y_n) \to d(x,y)$.

Assumption 1.19 :

$x_n \to x$ (ie; $d(x_n,x) \to 0$)and $x_n \leq y$ implies $x \leq y$

2 Main Results

2.1 Lemma :

Let $\{y_n\}$ be a sequence in X such that

$$\omega(y_{n+1},y_n) \leq \lambda\, \omega(y_n, y_{n-1}) \qquad \ldots(2.1.1)$$

where $0 < \lambda < 1$ then $\{y_n\}$is a Cauchy Sequence in X. Further, if $y_n \to y$ as $n \to \infty$, then $\omega(y,y) = 0$

Proof : We have by (2.1.1),

$\omega(y_{n+1}, y_n) \leq \lambda^n w(y_1, y_0)$, n = 1,2,3,…

$\therefore$ for n > m ,

$$\omega(y_n , y_m) \leq \omega(y_n, y_{n-1}) + \omega(y_{n-1}, y_{n-2}) + \omega(y_{n-2}, y_{n-3}) + \ldots + \omega(y_{m+1}, y_m)$$

$$\leq \lambda^{n-1} \omega(y_1, y_0) + \lambda^{n-2} \omega(y_1, y_0) + \ldots + \lambda^{m} \omega(y_1, y_0)$$

$$= [\lambda^{n-1} + \lambda^{n-2} + \lambda^{n-3} + \cdots + \lambda^{m}] \ \omega(y_1, y_0)$$

$$= \left(\frac{\lambda^m}{1-\lambda} \right) \omega(y_1, y_0) \qquad \ldots(2.1.2)$$

$$\to 0 \text{ as } m \to \infty$$

Now let 0 << η , choose δ according to (1.14).

By (2.1.2), $\omega(y_n , y_m) << \delta$ for m large

Hence $\omega(y_{n+1}, y_m) << \delta$ and $\omega(y_{n+1}, y_n) << \delta$

Hence $d(y_m , y_n) << \eta$ by (1.14)

Therefore $\{y_n\}$ is a Cauchy Sequence in (X,d)

Since $\omega(y_{n+1}, y_n) \leq \lambda^n \omega(y_1, y_0)$, n = 1,2,…

we get $\omega(y_{m+k} , y_m) \leq \lambda^m \omega(y_1, y_0)$ …(2.1.3)

for large m and all k

Hence it converges to y , say.

In (2.1.3) , letting $k \to \infty$ we get

$$\omega(y , y_m) \leq \lambda^m \ \omega(y_1, y_0)$$

Now letting $m \to \infty$, $\omega(y , y_m) \to 0$

Hence $\omega(y , y) = 0$

2.2 Lemma:

If $w(x, y) = 0$ and $w(y, x) = 0$ then

(i) $w(x, x) = w(y, y) = 0$ and

(ii) $d(x,y) = 0$ so that $x = y$

Proof : Since $\omega(x, x) \leq \omega(x, y) + \omega(y, x) = 0$

we get $\omega(x, x) = 0$. Similarly $\omega(y, y) = 0$

Also we have $\omega(x, y) = 0$ so that $d(x,y) = 0$

Hence $x = y$

2.3 Theorem:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,Ty)+\omega(Sy,Tx)+\omega(Sx,Tx)+\omega(Sy,Ty)+\max\{\omega(Tx,Ty),\omega(Ty,Tx)\}]$ …(2.3.1)

where $r \in [0,1/7)$ is a constant. Then S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof : Let $x_0 \in X$. Since S(X) is contained in T(X), we choose a point x_1 in X such that $S(x_0) = T(x_1)$. Continuing this process we choose x_n and x_{n+1} in X such that $S(x_n) = T(x_{n+1})$.

Then $\omega(Tx_{n+1}, Tx_n) = \omega(Sx_n, Sx_{n-1})$

$\leq r\,[\,\omega(Sx_n,Tx_{n-1})+\omega(Sx_{n-1},Tx_n)+\omega(Sx_n,Tx_n)+\omega(Sx_{n-1},Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$= r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}), \omega(Tx_{n-1},Tx_n)\}]$

$\therefore$ $w(Tx_{n+1}, Tx_n)$

$\leq r[\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}), \omega(Tx_{n-1},Tx_n)\}]$

Similarly $\omega(Tx_n, Tx_{n+1})$

$\leq r[\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}), \omega(Tx_{n-1},Tx_n)\}]$

$\therefore \max\{\omega(Tx_{n+1}, Tx_n), \omega(Tx_{n+1}, Tx_n)\}$

$\leq r[\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\leq r[\omega(Tx_{n+1},Tx_n) + \omega(Tx_n,Tx_{n-1}) + \omega(Tx_n,Tx_{n-1}) + \omega(Tx_{n-1},Tx_n) + \omega(Tx_{n+1},Tx_n)+$

$\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

Hence $\alpha_{n+1} \leq r[\alpha_{n+1} + \alpha_n + \alpha_n + \alpha_n + \alpha_{n+1} + \alpha_n + \alpha_n]$, where $\alpha_n = \max\{\omega(Tx_n,Tx_{n-1}), \omega(Tx_{n-1},Tx_n)\}$

$= r[2\alpha_{n+1} + 5\alpha_n]$

$\therefore$ $(1-2r)\alpha_{n+1} \leq 5r\alpha_n$

$\therefore$ $\alpha_{n+1} \leq \left(\frac{5r}{1-2r}\right)\alpha_n$

$\therefore$ $\alpha_n \to 0$ (since $\frac{5r}{1-2r} < 1$)

Hence $\{Tx_n\}$ is a Cauchy Sequence in (X,d).

Since T(X) is a totally closed subspace of X, there exists q in T(X) such that $Tx_n \to q$

as $n \to \infty$.

Cosequently we can find h in X such that $T(h) = q$. Thus

$\omega(Tx_n, Sh) = \omega(Sx_{n-1}, Sh)$

$\leq r[\omega(Sx_{n-1},Th)+\omega(Sh,Tx_{n-1})+\omega(Sx_{n-1},Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

$= r[\omega(Tx_n,Th)+\omega w(Sh,Tx_{n-1})+\omega(Tx_n,Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

On letting $n \to \infty$

$\omega(\text{Th},\text{Sh}) \leq r\,[\,\omega(\text{T}h,\text{T}h)+\omega(\text{S}h,\text{T}h)+\omega(\text{T}h,\text{T}h)+\omega(\text{S}h,\text{T}h)+\max\{\omega(\text{T}h,\text{T}h),\omega(\text{T}h,\text{T}h)\}]$

$= 2r\,\omega(\text{S}h,\text{T}h)$

$\therefore\ \omega(\text{Th},\text{Sh}) \leq 2r\,\omega(\text{S}h,\text{T}h)$.

Similarly $\omega(\text{Sh},\text{T}h) \leq 2r[\,\omega(\text{S}h,\text{T}h)\,]$

$\therefore\ \omega(\text{Th},\text{Sh}) = 0$ and $\omega(\text{S}h,\text{T}h) = 0$

$\therefore\ \text{S}h = \text{T}h$ (by Lemma 2.2)

Hence h is a coincidence point of S and T

Uniqueness : Suppose that there exists a point u in X such that $\text{S}u = \text{T}u$

So we have $\omega(\text{T}u,\text{T}h) = \omega(\text{S}u,\text{Sh})$

$\leq r\,[\,\omega(\text{S}u,\text{T}h)+\omega(\text{S}h,\text{T}u)+\omega(\text{S}u,\text{T}u)+\omega(\text{S}h,\text{T}h)+\max\{\omega(\text{T}u,\text{T}h),\omega(\text{T}h,\text{T}u)\}]$

$= r\,[\,\omega(\text{T}u,\text{T}h)+\omega(\text{T}h,\text{T}u)+\omega(\text{T}u,\text{T}u)+\omega(\text{T}h,\text{T}h)+\max\{\omega(\text{T}u,\text{T}h),\omega(\text{T}h,\text{T}u)\}]$

$\therefore\ \omega(\text{T}u,\text{T}h) \leq r\,[\,\omega(\text{T}u,\text{T}h) + \omega(\text{T}h,\text{T}u) + \max\{\omega(\text{T}u,\text{T}h),\omega(\text{T}h,\text{T}u)\}]$

Similarly $\omega(\text{T}h,\text{T}u) \leq r\,[\,\omega(\text{T}h,\text{Tu}) + \omega(\text{T}u,\text{T}h) + \max\{\omega(\text{T}h,\text{T}u),\omega(\text{Tu},\text{T}h)\}]$

$\therefore\ \max\{\omega(\text{T}u,\text{T}h),\omega(\text{T}h,\text{T}u)\} \leq r\,[\,\omega(\text{T}h,\text{Tu}) + \omega(\text{T}u,\text{T}h) + \max\{\omega(\text{T}h,\text{T}u),\omega(\text{Tu},\text{T}h)\}]$

Suppose $\omega(\text{T}u,\text{T}h) \leq r\,[\,\omega(\text{T}h,\text{Tu}) + \omega(\text{T}u,\text{T}h) + \omega(\text{Tu},\text{T}h)\}]$

so that $(1\text{-}2r)\,\omega(\text{T}u,\text{T}h) \leq r\,\omega(\text{T}h,\text{Tu})$

$\therefore\ \omega(\text{T}u,\text{T}h) \leq \frac{r}{1-2r}\,\omega(\text{T}h,\text{Tu})$

Similarly $\omega(\text{T}h,\text{T}u) \leq \frac{r}{1-2r}\,\omega(\text{T}u,\text{Th})$

$\leq \frac{r^2}{(1-2r)^2}\,\omega(\text{T}h,\text{Tu})$

$\therefore\ \omega(\text{T}h,\text{T}u) > \omega(\text{T}h,\text{T}u)$ (since $\frac{r}{1-2r} > 1$) , a contradiction

$\therefore\ \omega(\text{T}h,\text{T}u) = 0$

Similarly $\omega(\text{T}u,\text{T}h) >> \omega(\text{T}u,\text{T}h)$, a contradiction

$\therefore\ \omega(\,Tu, Th\,)\ = 0$

$\therefore\ Tu = Th$

Hence h is a unique coincidence point of S and T.

Now suppose that S and T are weakly compatible.

Then, by Proposition 1.17 **,** h is the unique common fixed point of S and T.

Assuming that T(X) is totally ordered , the following result of Sami Ullah Khan and Arjamand Bano [8] follows as a Corollary.

2.4 Corollary: (Theorem 3.2 **,** Sami Ullah Khan and Arjamand Bano [8])

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings S,T : X → X satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,Ty) + \omega(Sy,Tx) + \omega(Sx,Tx) + \omega(Sy,Ty) + \omega(Tx,Ty)\,]$ …(2.4.1)

where $r \in [0,{}^{1}/_{7}]$ is a constant. Then S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

2.5 Remark:

In theorem 2.3, if $T = I_X$, the identity map on X , then as a consequence of theorem 2.3, we obtain the following result.

2.6 Corollary:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings S,T : X → X satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\ [\ \omega(Sx,y)+\omega(Sy,x)+\omega(Sx,x)+\omega(Sy,y)+\max\{\omega(x,y),\omega(y,x)\}]$ …(2.6.1)

where $r \in [0,{}^{1}/_{7})$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

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