Fixed Point Theorems for Soft Compact Spaces, Pseudo-Soft Compact Tichonov Spaces

Sharad Kumar Barve[*1], R. D. Daheriya[2], Masroor Mohammad[3]

[1]*Research Scholar Department of Mathematics, J. H. Govt. P. G. College Betul, India-460001.*

[2]*Professor Department of Mathematics, J. H. Govt. P. G. College Betul, India-460001.*

[3]*Research Scholar Department of Mathematics Saifia Science College Bhopal, India-462001.*

Abstract:*In this paper, we prove some general soft fixed point theorems for self-mapping satisfying new contractive conditions in soft compact metric spaces and pseudo soft compact Tikhonov spaces, which in turn generalizes the result of Edelstein. (Edelstein, M. An extension of Banach's Contraction principal, Proc. Amer, Math, Soc. Vol. 12,7-10 (1961)).*

Keywords:*Fixed point, Contraction Mapping, Soft Compact Metric space.*

Introduction

Molodtsov [19] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties and has shown several applications of this theory in solving many practical problems in various disciplines like as economics, engineering, etc. Maji*et al.* [15, 16] studied soft set theory in detail and presented an application of soft sets in decision-making problems. Chen *et al.* [2] worked on a new definition of reduction and addition of parameters of soft sets, Shabir and Naz [21] studied about soft topological spaces and explained the concept of soft point by various techniques. Das and Samanta introduced a notion of soft real set and number [4], soft complex set and number [5], soft metric space [6, 7], soft normed linear space [8, 9]. Chiney and Samanta [3] introduced the concept of vector soft topology, Das, *et al.* [10] studied on soft linear space and soft normed space. By compact metric space X we mean any open covering of X has a finite sub-covering. If we have a collection of open sets which covers X, no matter whether it is a covering of a sub-covering, the collection of the complement of each open set has an empty intersection by DeMorgen's law and vice versa. So another formulation of compactness is that any collection of closed subsets of X with an empty intersection has a finite sub-collection also with an empty intersection. Furthermore, taking the contrapositive we have an equivalent definition of compactness: for any collection C of closed subsets of X, if every finite sub-collection of C has nonempty intersection, then the intersection of the sets in C is nonempty. We say a collection of sets of X has the finite intersection property if every finite sub-collection of X has nonempty intersection. So a concise version of the previous statement is

A space X is compact if and only if every collection of closed subsets of X with the finite intersection property has nonempty intersection.

Equipped with the above statement, we will find a fine connection between compactness and the

Bolzano-Weierstrass property. Recall that when we say a space X has the Bolzano-Weierstrass property, we mean any sequence in X has a cluster point (Is the cluster point in X?). In fact, compactness implies the Bolzano-Weierstrass property. To see why, let $\{x_n\}$ be a sequence in a compact space X. Then we consider the collection of closed sets $A = \{A_k\}_{k=1}^{\infty} = \overline{\{A_k\}_{k=1}^{\infty}}$ Clearly A has the finite intersection property. Because X is compact, we know that is nonempty. Let x be in the intersection. Let $O_x$be an open set containing x. For any $N, x \in A_N = \overline{\{x_N, x_{N+1}, \ldots\}}$. So there is $n \geq N$. So there is $x_n \in O_x$. This precisely says that x is a cluster point of $\{x_n\}$.

Recall that a space X is called sequentially compact if every sequence in it has a convergent sub-sequence. It can be readily checked that sequential compactness is the same as having the Bolzano-Weierstrass property (In the backward direction, we construct a subsequence converging to the cluster point).

In this paper, we are trying to find some fixed-point theorems for self-mapping on soft-compact metric spaces, which generalizes the results of Edelstein [2]. Also, we will find some soft fixed point theorems in pseudo soft compact tichonov spaces.

Definition 1.1 Soft fixed point space: Let T be a self-continuous mapping. A space $\tilde{Z}$ is called a fixed point space if every continuous mapping T of $\tilde{Z}$ into itself, has a soft fixed point, in the sense that $T(z_0) = z_0$

Definition 1.2A class $\{G_i\}$ of open subset of $\tilde{Z}$ is said to be an open cover of $\tilde{Z}$, if each point in $\tilde{Z}$ belongs to one G_i, i.e., $\cup_i G_i = \tilde{Z}$.

A subclass of an open cover which is at least an open cover is called a subcover.

A compact space is that space in which every open cover has a finite subcover. A subclass of an open cover which is at least an open cover is called a subcover. A compact space is that space in which every open cover has a finite subcover.

Definition1.3 Pseudo-compact tichonov spaces: A Topological space $\tilde{Z}$ is said to be pseudo-soft compact space, if every real valued continuous function on $\tilde{Z}$ is bounded. It may be noted that every soft-compact space is pseudo-soft compact but converse is not necessarily true. However, in a metric space notation 'compact' and 'pseudo compact' coincide. By Tichonov space we mean a completely regular Hausdroff space.

Theorem 1. 1 Let$T$ be a continuous mapping of a compact metric space $\tilde{Z}$ into itself satisfying theconditions;

$$d(T\tilde{z_1}, T\tilde{z_2}) \leq \alpha \frac{[d(\tilde{z_1}, T\tilde{z_1})d(\tilde{z_2}, T\tilde{z_1})] + [d(\tilde{z_2}, T\tilde{z_2})d(\tilde{z_1}, T\tilde{z_2})]}{d(\tilde{z_1}, T\tilde{z_1}) + d(\tilde{z_2}, \tilde{z_1}) + d(\tilde{z_2}, T\tilde{z_2}) + d(\tilde{z_1}, T\tilde{z_2})} + \beta \frac{[d(\tilde{z_1}, T\tilde{z_1})d(\tilde{z_2}, \widetilde{Tz_2})] + [d(\tilde{z_1}, T\tilde{z_2})d(\tilde{z_2}, T\tilde{z_1})]}{d(\tilde{z_1}, T\tilde{z_1}) + d(\tilde{z_2}, \widetilde{Tz_2}) + d(\tilde{z_1}, T\tilde{z_2}) + d(\tilde{z_2}, T\tilde{z_1})} + \gamma d(\tilde{z_1}, \tilde{z_2}) \quad (1.1)$$

$\forall \tilde{z_1}, \tilde{z_2} \in \tilde{Z}$ and where α, β, γ are nonnegative real numbers then T has a unique fixed point.

Proof: We define a function T on $\tilde{Z}$ as follows:

$S\tilde{z} = d(\tilde{z}, T\tilde{z}) \, \forall \, \tilde{z} \in \tilde{Z}$. Since d and T are continuous on $\tilde{Z}$, then T is also continuous on $\tilde{Z}$. From the compactness of $\tilde{Z}$, there exists a point $\tilde{A} \in \tilde{Z}$ such that

S($\tilde{A}$)=$inf\{S\tilde{z} : \tilde{z} \in \tilde{Z}\}$ (1.2)

If $S(\tilde{A}) \neq 0$, it follows that $\tilde{A} \neq T(\tilde{A})$ and so $S\left(T(\tilde{A})\right) = d(T(\tilde{A}), T^2(\tilde{A}))$

$$d(T\tilde{A}, T^2\tilde{A}) < \alpha \frac{[d(\tilde{A}, T\tilde{A})d(T\tilde{A}, T\tilde{A})] + [d(T\tilde{A}, T(T\tilde{A}))d(\tilde{A}, T(T\tilde{A}))]}{d(\tilde{A}, T\tilde{A}) + d(T\tilde{A}, T\tilde{A}) + d(T\tilde{A}, T(T\tilde{A})) + d(\tilde{A}, T(T\tilde{A}))} + \beta \frac{[d(\tilde{A}, T\tilde{A})d(T\tilde{A}, T(T\tilde{A}))] + [d(\tilde{A}, T(T\tilde{A}))d(T\tilde{A}, T\tilde{A})]}{d(\tilde{A}, T\tilde{A}) + d(T\tilde{A}, T(T\tilde{A})) + d(\tilde{A}, T(T\tilde{A})) + d(T\tilde{A}, T\tilde{A})} + \gamma d(\tilde{A}, T\tilde{A})$$

So $d(T\tilde{A}, T^2\tilde{A}) < \frac{\alpha}{2} d(T\tilde{A}, T^2\tilde{A}) + \frac{\beta}{2} d(T\tilde{A}, T^2\tilde{A}) + \gamma d(\tilde{A}, T\tilde{A})$

i.e., $\left(1 - \frac{\alpha}{2} - \frac{\beta}{2}\right) d(T\tilde{A}, T^2\tilde{A}) < \gamma d(\tilde{A}, T\tilde{A})$

i.e., $d(T\tilde{A}, T^2\tilde{A}) < \frac{\gamma}{1-\frac{\alpha}{2}-\frac{\beta}{2}} d(\tilde{A}, T\tilde{A})$

i.e., $S(T\tilde{A}) < RS\tilde{A}$ where $R = \frac{\gamma}{1-\frac{\alpha}{2}-\frac{\beta}{2}} \leq 1, \alpha + \beta + 2\gamma \leq 2$

which is contradiction to the condition (1.2) and hence $\tilde{A} = T\tilde{A}$ consequently, $\tilde{A}$ is a fixed point of T.

Uniqueness: Let $\tilde{B} \neq \tilde{A}$ be another soft fixed point of T.

Now $d(\tilde{A}, \tilde{B}) = d(T\tilde{A}, T\tilde{B})$

$$d(T\tilde{A}, T\tilde{B}) < \alpha \frac{[d(\tilde{A}, T\tilde{A})d(\tilde{B}, T\tilde{A})] + [d(\tilde{B}, T\tilde{B})d(\tilde{A}, T\tilde{B})]}{d(\tilde{A}, T\tilde{A}) + d(\tilde{B}, T\tilde{A}) + d(\tilde{B}, T\tilde{B}) + d(\tilde{A}, T\tilde{B})} + \beta \frac{[d(\tilde{A}, T\tilde{A})d(\tilde{B}, T\tilde{B})] + [d(\tilde{A}, T\tilde{B})d(\tilde{B}, T\tilde{A})]}{d(\tilde{A}, T\tilde{A}) + d(\tilde{B}, T\tilde{B}) + d(\tilde{A}, T\tilde{B}) + d(\tilde{B}, T\tilde{A})} + \gamma d(\tilde{A}, \tilde{B})$$

i.e., $d(\tilde{A}, \tilde{B}) < \left(\frac{\beta}{2} + \gamma\right) d(\tilde{A}, \tilde{B})$

which is a contradiction because $\beta + 2\gamma \leq 2$

Hence $\tilde{A}$ is a unique soft fixed point of T.

Theorem 2.1 Let P be a pseudo-soft compact Tichonov space and δ be a non-negative real valued continuous function over $P \times P$ satisfying;

$\delta(\tilde{z_1}, \tilde{z_1}) = 0, \tilde{z_1} \in P$

$$\delta(\tilde{z_1}, \tilde{z_2}) = \delta(\tilde{z_1}, \tilde{z_3}) + \delta(\tilde{z_3}, \tilde{z_2}) \forall \tilde{z_1}, \tilde{z_2}, \tilde{z_3} \in P. \qquad (2.1)$$

Let $T: P \times P$ be a continuous mapping satisfying for all distinct $\tilde{z_1}, \tilde{z_2} \in P$

$$\delta(T\tilde{z_1}, T\tilde{z_2}) < \delta(\tilde{z_1}, \tilde{z_2}) \left[1 + \frac{\delta(T\tilde{z_1}, \tilde{z_1})\delta(T\tilde{z_2}, \tilde{z_1})}{\delta(\tilde{z_1}, \tilde{z_2})} + \frac{\delta(T\tilde{z_1}, \tilde{z_1})\delta(T\tilde{z_2}, \tilde{z_1})}{\delta(T\tilde{z_1}, \tilde{z_1}) + \delta(T\tilde{z_2}, T\tilde{z_1})} + \frac{\delta(\tilde{z_2}, T\tilde{z_1})\delta(T\tilde{z_1}, \tilde{z_1})\delta(T\tilde{z_2}, \tilde{z_1})}{\delta(\tilde{z_2}, T\tilde{z_1}) + \delta(T\tilde{z_1}, \tilde{z_1}) + \delta(T\tilde{z_2}, \tilde{z_1})}\right] \qquad (2.2)$$

Then T has a unique soft fixed point.

Proof: We define a mapping by $d: \tilde{P} \rightarrow R$ by

$\psi(\tilde{p}) = \delta(T\tilde{p}, \tilde{p}), \tilde{p} \in \tilde{P}$.

where R is the set of real numbers clearly ψ is continuous, being the composite of two functions T and δ, since $\tilde{P}$ is pseudo-soft compact Tichonov space, hence every real valued continuous function over $\tilde{P}$ is bounded and attains its bounds. Thus there exists a point say $\tilde{v} \in \tilde{P}$, such that

$\psi(\tilde{v}) = inf\{\psi(\tilde{p}) : \tilde{p} \in \tilde{P}\}$.

It is clear that $\psi\ (\tilde{p}) \subset R$. We now affirm that $\tilde{v}$ is a soft fixed point for T. If not,

let us suppose that $T\tilde{v} \neq \tilde{v}$. So by (2.2)

$\psi(T\tilde{v}) = \delta(T^2\tilde{v}, T\tilde{v})$

$= \delta(T(T\tilde{v}), T\tilde{v})$

$$\psi(T\tilde{v}) < (T\tilde{v}, \tilde{v}) \left[1 + \frac{\delta(T(T\tilde{v}), T\tilde{v})\delta(T\tilde{v}, T\tilde{v})}{\delta(T\tilde{v}, \tilde{v})} + \frac{\delta(T(T\tilde{v}), T\tilde{v})\delta(T\tilde{v}, T\tilde{v})}{\delta(T(T\tilde{v}), T\tilde{v}) + \delta(T\tilde{v}, T\tilde{v})} + \frac{\delta(T(T\tilde{v}), T\tilde{v})\delta(T\tilde{v}, T\tilde{v})\delta(T(T\tilde{v}), \tilde{v})}{\delta(T(T\tilde{v}), T\tilde{v}) + \delta(T\tilde{v}, T\tilde{v}) + \delta(T(T\tilde{v}), \tilde{v})}\right] = \delta(T\tilde{v}, \tilde{v})$$

Which implies $\psi(T\tilde{v}) = \delta(T^2\tilde{v}, T\tilde{v}) < \delta(T\tilde{v}, \tilde{v})$

Hence a contradiction, so $T\tilde{v} = \tilde{v}$, i.e., $\tilde{v} \in \tilde{P}$ is a soft fixed point for T.

To prove the uniqueness of $\tilde{v}$, if possible, let $\tilde{w} \in \tilde{P}$be another fixed point for T, i.e.

$T\tilde{w} = \tilde{w}$ and $\tilde{w} \neq \tilde{v}$. So by (2.2).

$$\delta(\tilde{v}, \tilde{w}) = \delta(T\tilde{v}, T\tilde{w}) < \delta(\tilde{v}, \tilde{w}) \left[1 + \frac{\delta(T\tilde{v}, \tilde{v})\delta(T\tilde{w}, \tilde{v})}{\delta(\tilde{w}, \tilde{v})} + \frac{\delta(T\tilde{v}, \tilde{v})\delta(T\tilde{w}, \tilde{v})}{\delta(T\tilde{v}, \tilde{v}) + \delta(T\tilde{w}, \tilde{v})} + \frac{\delta(\tilde{w}, T\tilde{v})\delta(T\tilde{v}, \tilde{v})\delta(\tilde{v}, \tilde{w})}{\delta(\tilde{w}, T\tilde{v}) + \delta(T\tilde{v}, \tilde{v}) + \delta(\tilde{v}, \tilde{w})}\right] = \delta(T\tilde{v}, \tilde{v})$$

$\delta(\tilde{v}, \tilde{w}) < \delta(\tilde{v}, \tilde{w})$

Hence contradiction, thus $\tilde{v} = \tilde{w} \in \tilde{P}$ is unique soft fixed point for T in $\tilde{P}$.

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