A Note on Kevi Ring

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Abstract

In this paper, we introduce the concept of Kevi Ring. We will also give some properties and characterization of Kevi Ring. Examples are provided to illustrate our results.

Keywords : *Prime Ideal, Maximal Ideal, Local Ring, Kevi Ring, Nil Radical*

1. Introduction and Preliminaries

Throughout the paper, R is a commutative ring with unity. The concept of Prime ideals which arises in the theory of ring as a generalization of the concept of prime number in the ring of integers , plays a highly important role in that theory and it has been widely studied e.g. in [2] and [3]. The concept of ring having unique maximal ideal called local ring has been studied by M. F. Atiyah , I. G. MacDonal [1] and M. Nagata [4]. Now, we will introduce the concept of ring having unique prime ideal. We will use the notation η for nil radical of the ring.

We will also use the following definitions and results.

Definition 1.1 . An ideal $P \neq R$ of ring R is called **prime ideal** if $ab \in P$ implies either $a \in P$ or $b \in P \quad \forall\, a, b \in R$.

Definition 1.2 . An ideal $M \neq R$ of ring R is called **maximal ideal** if for any ideal I of R, $M \subseteq I \subseteq R$ implies either $I = M$ or $I = R$.

Definition 1.3 . A ring R which has unique maximal ideal is called **local ring** .

Definition 1.4 . The set of all nilpotent elements of ring R is called **nil radical** of ring .

Result 1.1 . Nil radical of the ring R is the intersection of all prime ideals of R.

Result 1.2 . Every maximal ideal is prime ideal.

Result 1.3 . Every non – unit element is contained in some maximal ideal.

Result 1.4 . Every field has only two ideals (0) and itself.

2. Results

We begin by introducing the following definition.

Definition 2.1 . A commutative ring with unity is called Kevi Ring if it has unique prime ideal.
e.g. $\mathbb{Z}_5$, $\mathbb{Z}_7$ etc. are kevi rings.

Theorem 2.1 . R is kevi ring iff every element of R is either unit or nilpotent.

Proof . Let R be a kevi ring.

Let $x \in R$ an element of R which is neither unit nor nilpotent .

$\therefore$ x is a non – unit element and every non – unit element is contained in some maximal ideal.

$\therefore$ $\exists$ maximal ideal M of R s.t. $x \in M$

Also, every maximal ideal is prime ideal.

$\therefore x \in M$ where M is prime ideal.

But R is kevi ring .

$\therefore$ it has unique prime ideal.

$\Rightarrow$ $x \in \eta$ (intersection of all prime ideals of R)

$\Rightarrow x$ is a nilpotent element.

which is contradiction

$\therefore$ every element of R is either unit or nilpotent .

Conversely .

Assume that every element of R is either unit or nilpotent.

Let P be a prime ideal.

$\therefore P$ can't contain unit element as if P contain unit element then $P = R$ which is contradiction to the definition of prime ideal .

$\therefore P$ contains all the nilpotent elements of the ring R as every element of ring is either unit or nilpotent.

$\therefore$ $P = \eta$

$\therefore R$ has unique prime ideal .

$\therefore R$ is kevi ring .

Theorem 2.2 . R is kevi ring iff R/η is field.

Proof . Let R be kevi ring.

$\therefore$ every element of R is either unit or nilpotent.

Let $\overline{x} \in R/\eta$ be non – zero element.

$\therefore x + \eta \neq 0 + \eta$

$\Rightarrow x \notin \eta$

$\therefore x$ is not a nilpotent element.

$\Rightarrow x$ is unit.

$\Rightarrow x + \eta$ is unit element of R/η .

$\therefore$ every non-zero element of R/η is unit.

$\Rightarrow R/\eta$ is field.

Conversely .

Let R/η is field.

we have to prove R is kevi ring i.e. it has unique prime ideal.

Suppose R has two distinct prime ideals P_1 and P_2 s.t.

$P_1 \neq R , P_2 \neq R$

$\therefore P_1/\eta$ and P_2/η are prime ideals of R/η.

But R/η is field and every field has only two ideals (0) and itself.

So, R has exactly one prime ideal.

$\therefore R$ is kevi ring.

Theorem 2.3 . Kevi ring has no idempotent other than 0 and 1.

Proof . Let R be kevi ring with P as its unique prime ideal.

Let e be idempotent element of R.

$\Rightarrow e^2 = e$

$\Rightarrow e(1 - e) = 0$

Case – I .

Suppose e is unit or $1 - e$ is unit.

I (i). If e is unit

$\Rightarrow 1 - e = 0$

$\Rightarrow e = 1$

I (ii). If $1 - e$ is unit

$\Rightarrow e = 0$

$\therefore$ either $e = 0$ or $e = 1$

Case – II .

Suppose e and $1 - e$ are non – unit.

Also every non – unit is contained in some maximal ideal and every maximal ideal is prime ideal.

$\therefore e$ and $1 - e$ are contained in some prime ideal.

But R has unique prime ideal ideal P.

$\therefore\ e, 1-e \in P$

$\Rightarrow e+1-e \in P$ ($\because P$ is an ideal)

$\Rightarrow 1 \in P$

$\Rightarrow P = R$

which is contradiction to the definition of prime ideal.

$\therefore$ kevi ring has no idempotent other than 0 and 1.

Theorem 2.4 . Every kevi ring is local ring.

Proof . Let R be a kevi ring but not local ring.

$\therefore R$ contains more than one maximal ideal.

But every maximal ideal is prime.

$\therefore R$ contains more than one prime ideal.

which is the contradiction to the definition of kevi ring.

$\therefore R$ is local ring.

$\Rightarrow$ Every kevi ring is local ring.

Note 2.1 . The converse of the above theorem is not true.
i.e. Local ring may not be kevi ring.
e.g. $\mathbb{Z}_4$ is local ring but it is not kevi ring.

References

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