Strong Coupled Fixed Point For Couplings in Multiplicative Metric Spaces

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Abstract. *The main purpose of this research article is to investigate the properties, existence and uniqueness of strong coupled fixed point in multiplicative metric space. This article is an analogue of Choudhury et al.*[5] *results in setting of multiplicative metric space. We have reported an example in support of our result.*

Keywords : Multiplicative Metric Space, Strong Coupled Fixed Point, Coupling

2010MSC : 47H10, 54H25,

1. Introduction and Mathematical Preliminaries

The concept of metric space was first introduced by Maurice Frèchet [10] in 1906 in his work Surquelques points du calcul fonctionnel. Later many researchers generalize and extend the concept of metric space among which multiplicative metric space was one of them. The concept of multiplicative metric space was given by Bashirov et al. [4] in 2008. The aim of introducing multiplicative metric space was to overcome the problem that the set of positive real numbers $\mathbb{R}$ is not complete with usual metric. The fixed point theory in the setting of multiplicative metric space has a vast literature such as [1, 2, 3, 7, 8, 11, 12]. On the other hand the concept of fixed point was generalized and coupled fixed point was introduced by T.G. Bhaskar and V. Lakshmikantham [6]. Later Choudhury et al. [5] introduced the concept of strong coupled fixed point and coupling and proved the existence and uniqueness of strong coupled fixed point for coupling in setting of metric spaces. In this paper we have proved the uniqueness of strong coupled fixed point in the setting of multiplicative metric space. We give an example to illustrate our results. Before going to the main result we recall some definitions.

Definition 1.1 [4] Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}$ satisfying the following axioms
(M1) $d(x, y) \geq 1, \forall x, y \in X, d(x, y) = 1 \Leftrightarrow x = y$;
(M2) $d(x, y) = d(y, x), \forall x, y \in X$;
(M3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

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Example 1.2 [4] Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined by

$$d(x,y) = \mid \frac{x_1}{y_1} \mid . \mid \frac{x_2}{y_2} \mid ... \mid \frac{x_n}{y_n} \mid,$$

where $X = (x_1, x_2, ..., x_n), Y = (y_1, y_2, ..., y_n), \in \mathbb{R}^n_+$ and $\mid . \mid : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined as

$$\mid a \mid = \begin{cases} a, & a \geq 1 \\ \frac{1}{a}, & a < 1. \end{cases}$$

Then (X, d) is a multiplicative metric space.

Definition 1.3 [9] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\epsilon(x) = \{y : d(x,y) < \epsilon\}, \epsilon > 1$, there exists a natural number $N \in \mathbb{N}$ such that $n \geq N$ and $x_n \in B_\epsilon(x)$. Then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x(n \to \infty)$.

Definition 1.4 [9] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

$$x_n \to x(n \to \infty) \Leftrightarrow d(x_n, x) \to 1(n \to \infty).$$

Definition 1.5 [9] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be multiplicative Cauchy sequence if, for $\epsilon > 1$, there exists a positive integer $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $n, m \geq N$.

Proportion 1.6 [9] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be multiplicative Cauchy sequence iff $d(x_n, x_m) \to 1(n, m \to \infty)$.

Definition 1.7 [9] A multiplicative metric space (X, d) is said to be multiplicative complete if every multiplicative Cauchy sequence in (X, d) is multiplicative convergent in X.

Definition 1.8 (Coupled Fixed Point) [6]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.9 (Strong Coupled Fixed Point) [5]. An element $(x, y) \in X \times X$, where X is any non-empty set, is called a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is coupled fixed point and $x = y$; that is if $F(x, x) = x$.

Definition 1.10 (Coupling) [5]. Let (X, d) be a metric space and A *and* B be two non-empty subsets of X. Then a function $F : X \times X \to X$ is said to be a coupling with respect to A *and* B if

$$F(x, y) \in B \text{ and } F(y, x) \in A.$$

whenever $x \in A$ and $y \in B$.

2. Main Result

Theorem 2.1 Let A and B be two non-empty closed subsets of a complete multiplicative metric space (X, d). Let $F : X \times X \to X$ be a coupling (with respect to A and B), *s.t.*

$$d(F(x, y), d(u, v)) \leq [max\{d(x, u), d(y, v)\}]^{\lambda}. \tag{1}$$

for any $x, v \in A$, $y, u \in B$ and $\lambda \in (0, \frac{1}{2})$, Then
(i) $A \cap B \neq \emptyset$,
(ii) F has a unique strong coupled fixed point in $A \cap B$.

Proof: Since A and B are non-empty subsets of X and F is a coupling, then for any $x_0 \in A$ and $y_0 \in B$ we define sequences $\{x_n\}$ and $\{y_n\}$ in A and B respectively by

$$x_{n+1} = F(y_n, x_n), \; y_{n+1} = F(x_n, y_n). \tag{2}$$

Suppose for some n, $x_n = y_{n+1}$ and $y_n = x_{n+1}$, Then by using 2, we get

$$x_n = y_{n+1} = F(x_n, y_n)$$

and

$$y_n = x_{n+1} = F(y_n, x_n)$$

.
Which shows that (x_n, y_n) is a coupled coincidence point of F.
We assume $x_n \neq y_{n+1}$ and $y_n \neq x_{n+1}$, $\forall \; n \in \mathbb{N}_0$.
We define a sequence $\{S_n\}$ by

$$S_n = max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\}. \tag{3}$$

We show $\lim_{n\to\infty} S_n = 1$. Now from 1, 2 and fact that $x_n \in A$ and $y_n \in B$, we get

$$\begin{aligned} d(x_1, y_2) &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\leq [max\{d(y_0, x_1), d(x_0, y_1)\}]^{\lambda}. \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(y_1, x_2) &= d(F(x_0, y_0), F(y_1, x_1)) \\ &\leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda}. \end{aligned} \tag{5}$$

From 4 and 5, we have

$$max\{d(x_1, y_2), d(y_1, x_2)\} \leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda}. \tag{6}$$

Again from 1, 2 and fact that $x_n \in A$ and $y_n \in B$, we get

$$\begin{aligned} d(x_2, y_3) &= d(F(y_1, x_1), F(x_2, y_2)) \\ &\leq [max\{d(y_1, x_2), d(x_1, y_2)\}]^{\lambda}. \end{aligned} \tag{7}$$

and

$$\begin{aligned} d(y_2, x_3) &= d(F(x_1, y_1), F(y_2, x_2)) \\ &\leq [max\{d(x_1, y_2), d(y_1, x_2)\}]^{\lambda}. \end{aligned} \tag{8}$$

using 6, 7 and 8, we get

$$\begin{aligned} max\{d(x_2, y_3), d(y_2, x_3)\} &\leq [max\{d(x_1, y_2), d(y_1, x_2)\}]^{\lambda} \\ &\leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^2}. \end{aligned} \tag{9}$$

We assume for some integer n,

$$d(x_n, y_{n+1}) \leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n}. \tag{10}$$

and

$$d(y_n, x_{n+1}) \leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n}. \tag{11}$$

Therefore from 10 and 11, we have

$$max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\} \leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n}. \tag{12}$$

Now we prove that 10 and 11 also holds for n+1, from 1, 2 and 10, we get

$$\begin{aligned} d(x_{n+1}, y_{n+2}) &= d(F(y_n, x_n), F(x_{n+1}, y_{n+1})) \\ &\leq [max\{d(y_n, x_{n+1}), d(x_n, y_{n+1})\}]^{\lambda} \\ &\leq [[max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n}]^{\lambda} \\ &= [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^{n+1}}. \end{aligned} \tag{13}$$

Similarly by using 1, 2 and 11, we get

$$d(y_{n+1}, x_{n+2}) \leq [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^{n+1}}. \tag{14}$$

Thus 13 and 14 shows that 10 and 11 also holds for n+1. Hence by the principle of mathematical induction we say that 10 and 11 holds $\forall\ n$.
Letting $n \to \infty$ in 12 and using the fact that $\lambda \in (0, \frac{1}{2})$, we have

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} max\{d(x_n, y_{n+1}), d(y_n, x_{n+1})\} \\ &\leq \lim_{n\to\infty} [max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n} \\ &= 1.\end{aligned}$$

Thus both

$$\lim_{n\to\infty} d(x_n, y_{n+1}) = 1 \ and \ \lim_{n\to\infty} d(y_n, x_{n+1}) = 1. \quad (15)$$

Again we define sequence $\{T_n\}$ by $T_n = d(x_n, y_n)$.
By using 1 and 2, we get

$$\begin{aligned}d(x_1, y_1) &= d(F(y_0, x_0), F(x_0, y_0)) \\ &\leq [max\{d(y_0, x_0), d(x_0, y_0)\}]^{\lambda} \\ &= [d(x_0, y_0)]^{\lambda}. \quad (16)\end{aligned}$$

Again by using 1, 2 and 16, we get

$$\begin{aligned}d(x_2, y_2) &= d(F(y_1, x_1), F(x_1, y_1)) \\ &\leq [max\{d(y_1, x_1), d(x_1, y_1)\}]^{\lambda} \\ &= [d(x_1, y_1)]^{\lambda} \\ &\leq [d(x_0, y_0)]^{\lambda^2}. \quad (17)\end{aligned}$$

We assume for some integer n,

$$d(x_n, y_n) \leq [d(x_0, y_0)]^{\lambda^n}. \quad (18)$$

Now by using 1, 2 and 18, we get

$$\begin{aligned}d(x_{n+1}, y_{n+1}) &= d(F(y_n, x_n), F(x_n, y_n)) \\ &\leq [max\{d(y_n, x_n), d(x_n, y_n)\}]^{\lambda} \\ &= [d(x_n, y_n)]^{\lambda} \\ &\leq [d(x_0, y_0)]^{\lambda^{n+1}}. \quad (19)\end{aligned}$$

This shows that 18 also holds for $n+1$. Thus by principle of mathematical induction we say that 18 holds $\forall\ n$.
As $\lambda \in (0, \frac{1}{2})$ and taking $n \to \infty$ in 18, we get

$$\lim_{n\to\infty} d(x_n, y_n) = 1. \quad (20)$$

By triangular inequality, 11 and 18, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_n).d(y_n, x_{n+1}) \\ &\leq [d(x_0, y_0)]^{\lambda^n}.[max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n} \\ &= [d(x_0, y_0).max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n} \\ &= K^{\lambda^n} \end{aligned} \tag{21}$$

where $K = d(x_0, y_0).max\{d(x_0, y_1), d(y_0, x_1)\}$.
Similarly by triangular inequality, 10 and 18, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d(y_n, x_n).d(x_n, y_{n+1}) \\ &\leq [d(x_0, y_0)]^{\lambda^n}.[max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n} \\ &= [d(x_0, y_0).max\{d(x_0, y_1), d(y_0, x_1)\}]^{\lambda^n} \\ &= K^{\lambda^n} \end{aligned} \tag{22}$$

where $K = d(x_0, y_0).max\{d(x_0, y_1), d(y_0, x_1)\}$.
Now we show $\{x_n\}$ and $\{y_n\}$ are multiplicative Cauchy sequences in A and B respectively.
For $m, n \in \mathbb{N}$ with $m \geq n$ and by using triangular inequality and 21, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}).d(x_{m-1}, x_{m-2})...d(x_{n+1}, x_n) \\ &\leq K^{\lambda^{m-1}}.K^{\lambda^{m-2}}...K^{\lambda^n} \\ &\leq K^{\frac{\lambda^n}{1-\lambda}}. \end{aligned} \tag{23}$$

As $\lambda \in (0, \frac{1}{2})$, therefore $d(x_m, x_n) \to 1$ as $(m, n \to \infty)$. This shows that $\{x_n\}$ is a multiplicative Cauchy sequence in A.
Similarly for $m, n \in \mathbb{N}$ with $m \geq n$ and by using triangular inequality and 22, we have

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m-1}).d(y_{m-1}, y_{m-2})...d(y_{n+1}, y_n) \\ &\leq K^{\lambda^{m-1}}.K^{\lambda^{m-2}}...K^{\lambda^n} \\ &\leq K^{\frac{\lambda^n}{1-\lambda}}. \end{aligned} \tag{24}$$

As $\lambda \in (0, \frac{1}{2})$, therefore $d(y_m, y_n) \to 1$ as $(m, n \to \infty)$. This shows that $\{y_n\}$ is a multiplicative Cauchy sequence in B.
Since A and B are closed in X which is complete multiplicative metric space, so A and B are complete in X. Therefore sequences $\{x_n\}$ and $\{y_n\}$ are multiplicative convergent in A and B respectively. Thus there exists $x \in A$ and $y \in B$, such that

$$x_n \to x \; and \; y_n \to y. \tag{25}$$

From 20, we get

$$x = y. \tag{26}$$

Thus $x = y \in A \cap B$, hence $A \cap B \neq \emptyset$ which proves part (i).
Now we show that $x \in A \cap B$ is the strong coupled fixed point of F.
From triangular inequality, 1, 2, 25 and 26, we get

$$\begin{aligned} d(x, F(x,x)) &\leq d(x, x_{n+1}).d(x_{n+1}, F(x,y)) \\ &= d(x, x_{n+1}).d(F(y_n, x_n), F(x,y)) \\ &\leq d(x, x_{n+1}).[max\{d(y_n, x), d(x_n, y)\}]^{\lambda} \\ &\to 1 \; as \; n \to \infty. \end{aligned}$$

$\Rightarrow F(x,x) = x$, hence we are done.

Uniqueness : Suppose if possible F has two strong coupled fixed points $l, m \in A \cap B$, then

$$F(l,l) = l \; and \; F(m,m) = m. \tag{27}$$

By using 1, 27 and fact that $l, m \in A \cap B$, we have

$$\begin{aligned} d(l,m) &= d(F(l,l), F(m,m)) \\ &\leq [max\{d(l,m), d(l,m)\}]^{\lambda} \\ &= [d(l,m)]^{\lambda}. \end{aligned}$$

Which is a contradiction as $\lambda \in (0, \frac{1}{2})$ and is only possible if $d(l,m) = 1 \; i.e. \; l = m$, which proves the uniqueness of strong coupled fixed point.

Example 2.2 Let $X = \mathbb{R}$, the set of real numbers and we define $d : X \times X \to X$ by $d(x,y) = e^{|x-y|}$, then we know that (X, d) forms a complete multiplicative metric space. Let $A = [0,2]$ and $B = [0,3]$ be the closed subsets of X. define $F : X \times X \to X$ by $F(x,y) = \lambda \mid x - y \mid, \forall \; x, y \in X$ and $\lambda \in (0, \frac{1}{2})$. Now we show all the conditions of above theorem are satisfied.
We first show that F is a coupling (with respect to A and B). For $x \in A$ and $y \in B$, we have

$$0 \leq F(x,y) \leq 3\lambda < \frac{3}{2}.$$

and

$$0 \leq F(y,x) \leq 3\lambda < \frac{3}{2}.$$

$\Rightarrow F(x,y) \in B$ and $F(y,x) \in A \; \forall \; x \in A$ and $y \in B$. Thus F is a coupling (with respect to A and B).
Now we show F satisfy the inequality (1), before proving this it should be noted that for any $a, b \in \mathbb{R}^+$, we have

$$e^{|a-b|} \leq max\{e^a, e^b\} \tag{28}$$

Also for any $a, b \in \mathbb{R}^+$ and $\lambda > 0$, we have

$$max\{a^\lambda, b^\lambda\} = (max\{a, b\})^\lambda. \tag{29}$$

Also we know that for any $a, b \in \mathbb{R}$, we have

$$| a - b | \geq || a | - | b || \tag{30}$$

Now by using 28, 29, 30 and the fact that $F(x, y) \geq 0$, $\forall x, y \in X$, we have for any $x, v \in A$, $y, u \in B$ and $\lambda \in (0, \frac{1}{2})$

$$\begin{aligned} d(F(x,y), F(u,v)) &= d(\lambda \mid x - y \mid, \lambda \mid u - v \mid) \\ &\leq e^{|\lambda|x-y|-\lambda|u-v||} \\ &= e^{\lambda||x-y|-|u-v||} \\ &\leq e^{\lambda|x-y-u+v|} \\ &= e^{\lambda|(x-u)-(y-v)|} \\ &\leq max\{e^{\lambda|(x-u)|}, e^{\lambda|(y-v)|}\} \\ &= [max\{e^{|(x-u)|}, e^{|(y-v)|}\}]^\lambda \\ &= [max\{d(x,u), d(y,v)\}]^\lambda. \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied.
Therefore $A \cap B$ is non-empty and F has a unique strong coupled fixed point in $A \cap B$ which is 0.

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