# Markovian queue with working vacation waiting server and set up times

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**Abstract**

In this paper, We consider the $M/M/1$ queue with a server working vacations and a waiting server with setup times. Using quasi birth and death process and matrix - geometric method, we give the distributions for the number of customers and the waiting time in the system in steady state. Furthermore, we obtain expected busy period. Finally, we get the stochastic decomposition structures of stationary indices.

**Keywords:**
M/M/1 queue, working vacation, setup times, Matrix Geometric Method.

## Introduction

In many real time situations, the server in the background queueing model many become unavailable for a random period of time to perform a secondary task, when there are no customers in the waiting time at the service completion epoch. Such period of server absence is termed as server vacation. Queueing models a subject to various polices are of interest to researchers in recent times owing to their widespread applicability. There are different types of vacation queueing systems. In the single vacation scheme, the server takes a vacation, of some random duration when the queue is empty. At the end of the vacation, the server returns to the queue. The server resumes service if there is at least one customer waiting upon his return from vacation. However, if the queue is empty on the server's return, the server waits to complete a busy period. In multiple vacation scheme, if the server returns from a vacation and finds the queue empty, he immediately commences another vacation. If there is at least one waiting waiting customer, then he will commence the service.

Queueing model subject to single or multiple exponential vacation are apt to model many practical scenarios[2,7,3].Vacation models are queueing systems in which servers may become unavailable for a period of time. In telecommunication systems, this period of absence may represent the server's working on some secondary job. In manufacturing systems, these unavailable periods may represent performing maintenance activities. Queues with server vacations have been extensively in the past, and have been successfully used in various applied problems. in the classical vacation queueing models, authors assumed the server is not available to serve customers during a vacation period. queueing systems with server vacations are manufacturing systems, and so on. A comprehensive and detailed review of the survey by Doshi [1]and Ke et al. [4]and the books by Takagi [10] and Tian and Zhang et al. [11,12]

Recall that in the multiple vacation queueing system it is assumed that the vacation times are independent and identically distributed. However, there are practical environments. Where this assumption may not be valid. Specifically, a vacation taken after a hard day's work during which many customers have been served may be longer than a vacation taken after the server returns from vacation and finds the queue empty. Servi and Finn [9]studied an M/M/1 queue. with multiple working vacations, and obtained the PGF of the number of customers in the system and the LST of waiting time distribution, and applied the results of performance analysis of gateway router in fiber communication networks. On the basis of Servi and Finn (2002), Liu et al [6] gave simple explicit expressions of distributions for the stationary queue length and waiting time and stochastic decomposition structures of stationary indices.

In this paper, we investigate an M/M/1 queue with a server vacations and a waiting server with setup times. Customers that see the off server waits at the server and the server is turned on. However, the server needs some setup time to be active so as to server the waiting customer.

The rest of this paper is organized as follows. In section 2 we establish the model. In section 3, we get the balance equations and derive the probability generating function of the number of customers. Various performance measures of this model are discussed in section 4. Section 5 presents some numerical examples. Finally, Section 6 concludes the paper.

**Model Formulation**

Consider an $M/M/1$ queueing system with first come first server discipline. customers arrive according to a Poisson process with rate $\lambda$. Upon the arrival of customers, if the server is free, arriving customers get service immediately. Customer service times are independently and identically distributed exponential random variables with parameter $\mu$. At the end of each busy period, the server waits for a random amount of time before proceeding on a working vacation. This random waiting time of the server is assumed to be exponentially distributed with a parameter $\eta_1$. The server begins a working vacation follows an exponential distribution with parameter $\gamma$. During a working vacation, arriving customers are served at a rate $\mu_v(\mu_v < \mu_b)$ according to arrival order. When a vacation ends, the server begins a closed-down period. During a closed-down period, an arriving customer cannot be served immediately and experiences a period of setup time, set-up duration follows an exponential distribution with parameter $\beta$ and a regular busy period starts after a busy period. We assume that inter arrival times, service times, vacation times and set-up times are mutually independent.

Let $N(t)$ denote the number of customers in the system at time t, and let J(t) be the state of server at time t. There are three possible states of the server as follows.

$J(t)$= the server is in a working vacation period at time $t$ and the server is free, 1, the server is during a normal service period at time $t$ and the server is busy , 2, the server is in a set-up period or closed-down period at time $t$ and the server is free.

Then $(J(t), N(t)), t \geq 0$ is a Markov process with the state space, $\Omega = \{(0,j), j = 0,1,2\} \cup \{(k,j), j \geq 1, k = 0,1,2\}$

Using the Lexicographical sequence for the states, the infinitesimal generator can be written as

$$Q=\begin{pmatrix} B_{00} & C_0 \cdots & & & \\ B_0 & A_0 & C_0 \cdots & & \\ & B_0 & A_0 & C_0 \cdots & \\ & & B_0 & A_0 & C_0 \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

where

$$B_{00} = \begin{pmatrix} -(\lambda+\gamma) & \gamma & 0 \\ 0 & -(\lambda+\beta) & \beta \\ \eta_1 & 0 & -(\lambda+\eta_1) \end{pmatrix}, \quad C_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$B_0 = \begin{pmatrix} \mu_v & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_b \end{pmatrix}, \quad A_0 = \begin{pmatrix} -(\lambda+\gamma+\mu_v) & 0 & \theta \\ 0 & -(\lambda+\beta) & \beta \\ 0 & 0 & -(\lambda+\mu_b) \end{pmatrix}$$

Due to the block structure of matrix Q, $(Q(t), J(t)), t \geq 1$ is called a QBD process.

**Theorem**

The QBD process $(Q(t), J(t)), t \geq 1$ is positive recurrent if and only if $\lambda\beta(\mu)^{-1} < 1$

**proof**

$$C_{00} = B_0 + A_0 + C_0 = \begin{pmatrix} -\gamma & 0 & \gamma \\ 0 & -\beta & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

Since matrix $C_{00}$ is reducible, the theorem 7.3.1 in [5] gives the condition for positive recurrence of the QBD. After permutation of rows and columns. The theorem 7.3.1 states that the QBD is positive recurrent if and only if

$V\begin{pmatrix} 0 & 0 \\ 0 & \mu_b \end{pmatrix} i > V\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} i$

$V\begin{pmatrix} -\beta & 0 \\ 0 & \beta \end{pmatrix} = 0, Vi = 1$

the QBD process is positive recurrent if and only if $\mu_b > \lambda$ The structure of $\tilde{R}$ indicates that $\{R(t), C(t)\}$ is a quasi birth death process,see Neuts [8] or Latouche and Ramaswam[5]. To analyze this QBD process, it is necessary to solve for the minimal non-negative solution of the matrix quadratic equation

$$R^2B_0 + RA_0 + C = 0 \qquad (1)$$

and this solution is called the rate matrix and denoted by R.

**Lemma1:**

If $\lambda < \mu$, the matrix equation $R^2B_0 + RA_0 + C_0 = 0$ has the minimal non-negative solution

$$R = \begin{pmatrix} r & 0 & \frac{\gamma r}{\mu_b(1-r)} \\ 0 & \theta & \mu \\ 0 & 0 & \rho \end{pmatrix} \qquad (2)$$

**Proof:**

Since the matrices $A, B, C$ of $R^2B_0 + RA_0 + C_0 = 0$ are all upper triangular, we can assume that R has the same structure as

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

substituting $R^2$ and R into (1), we get the following set of equations.

$$\begin{cases} \mu_v r_{11}^2 - (\lambda + \gamma + \mu_v) r_{11} + \lambda = 0 \\ -r_{12}(\lambda + \beta) = 0 \\ \mu_b(r_{11}r_{13} + r_{12}r_{23} + r_{13}r_{33}) + r_{11}\gamma + \beta r_{12} - r_{13}(\lambda + \mu_b) = 0 \\ \lambda - r_{22}(\lambda + \beta) = 0 \\ r_{23}\mu_b(r_{22} + r_{33}) + \beta r_{22} - r_{23}(\lambda + \mu_b) = 0 \\ \mu_b r_{33}^2 - r_{33}(\lambda + \mu_b) + \lambda = 0 \end{cases} \qquad (3)$$

To obtain the minimal non-negative solution of (1),by using the equation

$$r_{11} = \frac{(\lambda + \gamma + \mu_v) - \sqrt{(\lambda + \gamma + \mu_v)^2 - 4\mu_v\lambda}}{2\mu_v}$$

(the other root is>1)in the first equation of (3).

$r_{22} = \frac{\lambda}{\lambda+\beta} = \alpha_2$ in the fourth equation of $(3)$, $r_{33} = \rho$ in the last equation of $(3)$, $r_{12} = 0$ in the second equation of $(3)$, $r_{23} = \rho$ in the fifth equation of $(3)$, weget $r_{13} = \frac{\gamma r}{\mu_h(1-r)}$. Thus, we get equation of $(2)$.

Because r satisfies the following equation

$\mu_v r^2 - (\lambda + \gamma + \mu_v)r + \lambda = 0$

$\lambda + \gamma + \mu_v(1 - r) = \frac{\lambda}{r}$,

equivalently, we have

$\frac{\gamma}{1-r} + \mu_v = \frac{\lambda}{r}$ which completes the proof.

**Theorem:** The QBD process $\{Q(t), J(t), t > 1\}$ is positive recurrent if and only if $\rho < 1$.

**Proof.** Based on the Theorem Neuts $(1981)$[8], the QBD process $(Q(t), J(t)); t \geq 0$ is positive recurrent if and only if the spectral radius $SP(R) < 1$, R is denoted by rate matrix and set of equations $(x_0, x_1, x_2, x_3, x_4, x_5)B[R] = 0$ has positive solution.

$$B[R] = \begin{pmatrix} B_{00} & C_0 \\ B_0 & RB_0 + A_0 \end{pmatrix} \tag{4}$$

$$B[R] = \begin{pmatrix} -(\lambda+\gamma) & \gamma & 0 & \lambda & 0 & 0 \\ 0 & -(\lambda+\beta) & \beta & 0 & \lambda & 0 \\ \eta_1 & 0 & -(\lambda+\eta_1) & 0 & 0 & \lambda \\ \mu_v & 0 & 0 & -\frac{\lambda}{r} & 0 & \frac{\gamma}{1-r} \\ 0 & 0 & 0 & 0 & -(\lambda+\beta) & (\lambda+\beta) \\ 0 & 0 & \mu_b & 0 & 0 & -\mu_b \end{pmatrix}$$

$B[R]$ is an irreducible and aperiodic generator with finite state.

Therefore$(x_1, x_2, X_3, X_4, x_5)B(R) = 0$ has positive solution.

Thus,process $\{(Q(t), J(t)); t \geq 0\}$ is positive recurrent if and only if $SP(R) = max(r, \alpha, \rho) < 1$. note that $0 < \alpha_1 < 1$, and $0 < \alpha_2 < 1$ the above relation means that $\rho < 1$.

**Queue length distribution**

If $\lambda < \mu_b$, let$(Q, J)$ be the stationary limit of the $QBD$ process $\{(Q(t), C(t)); t \geq 0\}$.

Denote k=0 $, (\pi_{0k}, \pi_{1k}, \pi_{2k})$

$\pi_{kj} = P\{Q = k, J = j, t \geq 0\} = t \xrightarrow{lt} \infty\{Q(t) = k, C(t) = j, t \geq 0\}, (k, j) \in \omega.$

Using quasi birth and death process and the matrix-geometric solution method,it is easy to get the following theorem.

**Theorem:**

If $\lambda < \mu_b$, the stationary probability distribution of $(Q(t), J(t); t \geq 0)$ is

$$\begin{cases} \pi_{0k} = \pi_{00} r^k, k \geq 0, \\ \pi_{1k} = \gamma\theta^k \pi_{00}, k \geq 0 \\ \pi_{2k} = \pi_{00}\left[\frac{\gamma r}{\mu_b(1-r)}\sum_{j=1}^{k-1} \rho^j \rho^{k-1-j} + \frac{\gamma\lambda}{(\lambda+\beta)^2}\sum_{j=1}^{k-1} \rho^j \theta^{k-1-j} \right. \\ \left. + \frac{1}{\mu_h}\left(\frac{(\lambda+\eta_1)(\lambda+\gamma-\mu_v r)}{\eta_1} - \frac{\gamma\beta}{\lambda+\beta}\right)\rho^{k-1}\right], \qquad k \geq 1. \end{cases} \tag{5}$$

where

$$\pi_{00}=[\frac{1}{1-r}+\frac{\gamma}{\lambda+\beta}+\left(\frac{1}{\rho(1-\rho)\mu_b(1-r)^2}\left(\gamma r(r+\rho-1)+(1-r)^2((\lambda+\eta_1(\lambda+\gamma-\mu_v r)-\frac{\lambda\beta}{\lambda+\beta})\right)+\frac{\gamma\lambda}{(\lambda+\beta)^2}\frac{\rho+\theta-1}{\theta(1-\rho)(1-\theta)}\right)]^{-1}.$$

**Proof.**

With the matrix-geometric solution method see[5]or[7], we have

$$\pi_k-(\pi_{0k},\pi_{1k},\pi_{2k})-(\pi_{01},\pi_{11},\pi_{21})R^{k-1},k\geq 1. \quad (6)$$

and $(\pi_{00},\pi_{10},\pi_{20},\ \pi_{01},\ \pi_{11},\pi_{21}$ satisfies the set of equations $(\pi_{00},\pi_{01},\pi_{10},\pi_{11},\pi_{12})B[R]=0$ substituting $B[R]$ in $(5)$ into the above relation, we obtain

$$\begin{cases}-(\lambda+\gamma)\pi_{00}+\eta_1\pi_{20}+\mu_v\pi_{01}=0\\ \gamma\pi_{00}-(\lambda+\beta)\pi_{10}=0\\ \beta\pi_{10}-(\lambda+\eta_1)\pi_{20}+\mu_b\pi_{21}=0\\ \lambda\pi_{00}-\frac{\lambda}{r}\pi_{01}-0\\ \lambda\pi_{10}-(\lambda+\beta)\pi_{11}=0\\ \lambda\pi_{20}+\frac{\gamma}{1-r}\pi_{01}+(\lambda+\beta)\pi_{11}-\mu_b\pi_{21}\end{cases}$$

asking $\pi_{00}$ as a known constant, we get

$$(\pi_{00},\pi_{10},\pi_{20},\pi_{01},\pi_{11},\pi_{21})=\pi_{00}\left(1,\frac{\gamma}{\lambda+\beta},\frac{\lambda+\gamma-\mu_v r}{\eta_1},r,\frac{\lambda\gamma}{(\lambda+\beta)^2},\frac{1}{\mu_b}(\frac{\lambda+\eta_1}{\eta_1}(\lambda+\gamma-\mu_v r)-\frac{\gamma\beta}{\lambda-\beta})\right). \quad (7)$$

and $R^k=\left(r^k \quad 0 \quad \frac{\gamma r}{\mu_b(1-r)}\sum_{j=1}^{k-1} r^j\rho^{k-1-j} 0 \quad a_2^k \quad \sum_{j=1}^{k-1}\rho^j\theta^{k-1-j} 0 \quad 0 \quad \rho_k\right), k\geq 1$

Substituting $(\pi_{01},\pi_{11},\pi_{21})$ and $R^{k-1}$ into (6), we obtain (5)

Finally, $\pi_{00}$ can be determined by the normalization condition.

with(6), the probabilities of the server in various state are as follows, respectively

P{ J=0 } $=\sum_{k=0}^{\infty}\pi_{0k}=\pi_{00}\frac{1}{1-r}$

P{ the server is in a Closed-down period }$=\pi_{10}=\pi_{00}\frac{\gamma}{\lambda+\beta}$,

P{ the server is in a set-up period } $=\sum_{k=0}^{\infty}\pi_{1k}=\pi_{00}\frac{\gamma\lambda}{\beta}$,

P{the server is in regular service period} $=\sum_{k=0}^{\infty}\pi_{2k}$

$$=\pi_{00}[\frac{\gamma r}{\mu_b(1-r)}\sum_{k=0}^{\infty}\sum_{j=1}^{k-1} r^j\rho^{k-1-j}+\frac{\gamma\lambda}{(\lambda+\beta)^2}\sum_{k=0}^{\infty}\sum_{j=1}^{k-1}\rho^j\theta^{k-1-j}\frac{1}{\mu_b}(\frac{(\lambda+\eta_1)(\lambda+\gamma-\mu_v r)}{\eta_1}-\frac{\gamma\beta}{\lambda+\beta})\sum_{k=0}^{\infty}\rho^{k-1}]$$

$$=\pi_{00}[\frac{1}{\rho(1-\rho)\mu_b(1-r)^2}(\gamma r(r+\rho-1)+(1-r)^2((\lambda+\eta_1)(\lambda+\gamma-\mu+vr)-+\frac{\gamma\lambda(\rho+\theta-1)}{(\lambda+\beta)^2\theta(1-\rho)(1-\theta)}]$$

The constant factor $\pi_{00}$ can be determined by the normalization condition.

$$\pi_{00}=[\frac{1}{1-r}+\frac{\gamma}{\lambda+\beta}\frac{1}{\rho(1-\rho)\mu_b(1-r)^2}(\gamma r(r+\rho-1)+(1-r)^2((\lambda+\eta_1)(\lambda+\gamma-\mu+vr)-\frac{\lambda\beta}{\lambda+\beta}))+\frac{\gamma\lambda(\rho+\theta-1)}{(\lambda+\beta)^2\theta(1-\rho)(1-\theta)}]^{-1}.$$

Now, We give stochastic decomposition structure of the number of customers Q in system in steady-state results.

**Theorem** If $\lambda<\mu$ the number of customers Q in system can be decomposed into the sum of two independent random variables: $Q=Q_0+Q_d$, where $Q_0$ is the number of customers of a classic $M/M/1$ queue in steady-state and follows a geometric distribution with parameter $1-\rho$; Additional number of customers $Q_d$ has a modified geometric distribution

$$P\{Q_d = k\} = \begin{cases} \pi_{00}\omega_1, k = 0 \\ \pi_{00}\omega_2, k = 1 \\ \pi_{00}\omega_3(1-r)r^{k-1} + \pi_{00}r_4(1-\theta)\theta^{k-1}, k \geq 2 \end{cases}$$

where

$$\theta = \frac{\lambda}{\lambda+\beta}$$

$$\Omega_1 = (1-\theta)(1-r) + \frac{1}{\lambda}\left(\frac{(\lambda+\eta_1)}{\eta_1}(\lambda+\gamma-\mu_v r) - \frac{\gamma\beta}{\lambda+\beta}\right)(1-r)(1-\theta),$$

$$\Omega_2 = \left[\left((r-\rho) + \frac{\gamma r}{\mu_b(1-r)}\right) + \left(\frac{\theta-\rho}{\lambda+\beta} + \frac{\theta}{\mu_b}\right)\gamma\right](1-r)(1-\theta),$$

$$\Omega_3 = \left[(r-\rho) + \frac{\gamma r}{\mu_b(1-r)}\right](1-\theta),$$

$$\omega_4 = \left[\frac{\theta-\rho}{\lambda+\beta} + \frac{\theta}{\mu_b}\right]\gamma(1-r).$$

$$K = \frac{\pi_{00}}{(1-\rho)(1-\beta)(1-\theta)}$$

**Proof.** With (5), the probability generating function of Q can be written as

$$\begin{aligned} Q(z) &= \sum_{k=0}^{\infty}\pi_{0k}z^k + \sum_{k=0}^{\infty}\pi_{1k}z^k + \sum_{k=0}^{\infty}\pi_{2k}z^k \\ &= \pi_{00}\Big[\frac{1}{1-rz} + \frac{\gamma}{(\lambda+\beta)(1-6z)} + \frac{\gamma r}{\mu_b(1-r)}\frac{(\rho+r)z-1}{\rho(1-rz)(1-\rho z)} + \frac{\gamma\lambda((\theta+\rho)z-1)}{(\lambda+\beta)^2\theta(1-\rho z)(1-\theta z)} \\ &\quad + \frac{1}{\mu_b}\left[\frac{(\lambda+\eta_1)(\lambda+\gamma-\mu_v r)}{\eta_1} - \frac{\gamma\beta}{\lambda+\beta}\right]\frac{1}{\rho(1-\rho z)}\Big] \end{aligned}$$

$$\begin{aligned} Q(z) &= \frac{(1-\rho)}{1-\rho z}K\Big[\frac{(1-r)(1-\theta)(1-\rho z)}{1-rz} + \frac{\gamma(1-r)(1-\theta)(1-\rho z)}{(1-6z)(\lambda+\beta)} \\ &\quad + \frac{1}{\mu_b}\left[\frac{(\lambda+\eta_1)(\lambda+\gamma-\mu_v r)}{\eta_1} - \frac{\gamma\beta}{\lambda+\beta}\right]\frac{(1-r)(1-\theta)}{\rho} \\ &\quad + \frac{\gamma r}{\lambda(1-r)}(\rho+r)(1-\theta)\frac{1-r}{1-rz}z - \frac{\gamma r}{\lambda(1-r)}\frac{(1-r)}{1-rz}(1-\theta) \\ &\quad + \frac{\gamma r}{\lambda+\beta}(1-r)(\theta+\rho)\frac{1-\theta}{1-\theta z}z - \frac{\gamma 1-r}{\lambda+\beta}\frac{(1-\theta)}{1-\theta z}\Big] \end{aligned}$$

$$\frac{1-r}{1-rz}(1-\rho z) = (1-r) + (r-\rho)\frac{(1-r)z}{1-rz}$$

$$\frac{1-\theta}{1-\theta z}(1-\rho z) = (1-\theta) + (\theta-\rho)\frac{1-\theta}{1-\theta z}z$$

$$Q(z) = \frac{1-\rho}{1-\rho z}K\left[\Omega_1 + \Omega_2 z + \Omega_3\frac{r(1-r)}{1-rz}z^2 + \Omega_4\frac{\theta(1-\theta)}{1-\theta z}z^2\right]$$

where, $\Omega_1 = (1-\theta)(1-r) + \frac{1}{\lambda}\left(\frac{(\lambda+\eta_1)}{\eta_1}(\lambda+\gamma-\mu_v r) - \frac{\gamma\beta}{\lambda+\beta}\right)(1-r)(1-\theta),$

$$\Omega_2 = \left[\left((r-\rho) + \frac{\gamma r}{\mu_b(1-r)}\right) + \left(\frac{\theta-\rho}{\lambda+\beta} + \frac{\theta}{\mu_b}\right)\gamma\right](1-r)(1-\theta),$$

$$\Omega_3 = \left[(r-\rho) + \frac{\gamma r}{\mu_b(1-r)}\right](1-\theta),$$

$$\omega_4 = \left[\frac{\theta-\rho}{\lambda\ \beta} + \frac{\theta}{\mu_b}\right]\gamma(1-r)$$

$$= \frac{1-\rho}{1-\rho z}Q_d(z)$$

(7)

**Remark:**

$$E(Q_d) = k\left[\omega_2 + \frac{2r-r^2}{1-r}\omega_3 + \frac{2\theta-\theta^2}{1-\theta}\omega_4\right]E(Q) = \frac{\rho}{1-\rho} + E(Q_d)$$

**Waiting Time Distribution**

Denote the Waiting time analysis is very important in practical model. Denoted the waiting time of a customer in the system by W, we can get the following stochastic decomposition results.

**Theorem:** If $\lambda < \mu$, the virtual time W can be decomposed into the sum of two independent random variables: $W = W_0 + W_d$, where $W_0$ is the virtual time of a customer in a corresponding classic M/M/1 queue and has an exponential distribution with parameter $\mu(1-\rho)$; Additional delay $W_d$ has the LST

$$W_d^*(s) = k[\gamma_1 + \gamma_2 \frac{\sigma}{\sigma+s} + \gamma_3 \frac{\beta}{\beta+s}].$$

where,

$$\sigma = \frac{\lambda(1-r)}{r}$$

$$\gamma_1 = \omega_1 + \omega_2 - \frac{1-r^2}{r}\omega_3 - \frac{(1-\theta)(2\lambda+\beta)}{\lambda}\omega_4$$

$$\gamma_2 = \frac{1}{r}\omega_3$$

$$\gamma_3 = \frac{1}{\theta}\omega_4$$

**Proof.** The classical relation between the PGF of Q and the LST of Waiting time W is $Q(z) = W^*(\lambda(1-z))$ From Theorem 3, the PGF of the number of customers Q Can be written as

$$Q(Z) - \frac{(1-\rho)}{1-\rho z}K\left[\omega_1 + \omega_2 z + \omega_3 \frac{r(1-r)}{1-rz}z^2 + \omega_4 \frac{\theta(1-\theta)}{1-\theta z}z^2\right]$$

Taking $z = 1 - \frac{s}{\lambda}$ in above equation, denote $\frac{\lambda(1-r)}{r} = \sigma$ note that

$$\frac{1-\rho}{1-\rho z}|_{z=1-\frac{s}{\lambda}} - \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+s}$$

$$\frac{z^2}{(\lambda+\beta)-\lambda z}|_{z=1-\frac{s}{\lambda}} = \frac{(1-\frac{s}{\lambda})^2}{\beta+s} = \frac{1}{\lambda^2}\left[\frac{(\lambda+\beta)^2}{\beta+s} - (2\lambda+\beta)+s\right]$$

$$\frac{z^2}{1-rz}|_{z=1-\frac{s}{\lambda}} = \frac{(1-\frac{s}{\lambda})^2}{(1-r)+\frac{rs}{\lambda}} = \frac{1}{\lambda r}\left[\frac{(\frac{\lambda}{r})^2}{\sigma+s} - \sigma + s\right]$$

Substituting the above results into (*), we have

$$W^*(s) = \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+s}K[\omega_1 + \omega_2(1-\frac{s}{\lambda}) + \omega_3\frac{1-r}{\lambda}\left[\frac{(\frac{\lambda}{r})^2}{\sigma+s} - \sigma + s\right] + \omega_4\frac{1-\theta}{\lambda}\left[\frac{(\lambda+\beta)^2}{\beta+s} - (2\lambda+\beta)+s\right]]$$

$$W^*(S) = \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+S}W_d^*(s),$$

Therefore, $W_d^* s$ is a Laplace Stieltjes transform.

**Remark:**

$$E(w_d) = k\left(\gamma_2\frac{1}{\sigma} + \gamma_3\frac{1}{\beta}\right)$$

$$E(w_d) = \frac{1}{\lambda}E(Q_d)$$

$$E(W) = \frac{1}{\mu_b(1-\rho)} + E(W_d)$$

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