A pathway to matrix–variate Gamma and Normal densities in complex case

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ABSTRACT

A general complex matrix–variate probability model is introduced here, which covers almost all complex matrix-variate densities used in multivariate statistical analysis. Through the new density introduced here, a pathway is created to go from matrix-variate type-1 beta to matrix-variate type-2 beta to matrix-variate gamma to matrix-variate Gaussian or normal densities in complex case. Connections to the distributions of quadratic forms and generalized quadratic forms in the new matrix are established. The present day analysis of these problems is mainly confined to Gaussian random variables. Thus, through the new distribution, all these theories are extended. Connections to certain geometrical probability problems, such as the distribution of the volume of a random paralleltope in Euclidean space, is also established.

KEYWORDS: *Function of matrix arguments; t, F, Cauchy distributions; Hermitian positive definite; Quadratic forms.*

1) INTRODUCTION:

Function of matrix argument in the complex case: We consider real valued scalar function of a single matrix argument of the type $\tilde{Z} = \tilde{X} + i\tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are p x p matrices with real elements and $i = \sqrt{-1}$ as well as scalar functions of many matrices $\tilde{Z}_j$, j = 1, 2, …….K where each $\tilde{Z}_j$ is of the type $\tilde{Z}$ above in the real case. We confined our discussion to the situation where the argument matrix was real symmetric positive definite. This was done so that the fractional power of matrices and functions of such matrices could be uniquely defined. Corresponding properties are of we restrict to the class of Hermitian positive definite matrices.

Definition: Hermitian positive definite matrix due to Mathai [12], We will denote the conjugate of $\tilde{Z}$ by $\tilde{\tilde{Z}}$ if $\tilde{Z}$ Hermitian, then $\tilde{Z} = \tilde{Z}*$, that is

$$\tilde{Z} = \tilde{Z}* \quad \Rightarrow \quad \tilde{X} + i\tilde{Y} = (\tilde{X} + i\tilde{Y})* = \tilde{X}' + i\tilde{Y}'$$

$$\Rightarrow \quad \tilde{X} = \tilde{X}' \text{ and } \tilde{Y} = \tilde{Y}'$$

Thus $\tilde{X}$ is the symmetric and $\tilde{Y}$ is skew symmetric. Further if $\tilde{Z}$ is Hermitian positive definite, then all the Eigen values of $\tilde{Z}$ are real and positive. Further, matrix variate gamma in the complex case is

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}} \Gamma(\alpha)\Gamma(\alpha-1)......\Gamma(\alpha-\pi+1)$$

We will use the notation $\tilde{Z} > 0$ to indicate that $\tilde{Z}$ is Hermitian positive definite. Constant matrices will be written without a tilde whether the elements are real or complex unless it has to be emphasized that the matrix involved has complex elements. Then in that case a constant matrix will also be written with a tilde.

Let $\tilde{X} = (x_{ij}), i = 1,...p, j = 1,...r, r \geq p,$ of rank p and of complex variables x_{ij}'s for all i and j, subject to the condition that the rank of $\tilde{X}$ is p, having the density $f(\tilde{X})$, where $f(\tilde{X})$ is a scalar function of $\tilde{X}$ is given by

$$f(\tilde{X}) = c\left|\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^{\alpha}\left|I - a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^{\frac{\beta}{1-q}} \quad ...(1.1)$$

for $\tilde{A} = \tilde{A}' > 0$ *and* $p \times p$, $\tilde{B} = \tilde{B}' > 0$ *and* $r \times r, a, \beta, q$ scalars, $a > 0, \beta > 0,$

$I - a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}} > 0,$ where $\tilde{A}$ *and* $\tilde{B}$ are free of the elements in $\tilde{X}$ and c is the normalizing constant. For convenience let $\tilde{A}^{\frac{1}{2}}$ *and* $\tilde{B}^{\frac{1}{2}}$ denote the symmetric positive definite square roots of $\tilde{A}$ *and* $\tilde{B}$ respectively. $\tilde{A}$ prime denotes the transpose, $|(\bullet)|$ denotes the determinant of (.) , I is the identity matrix, (.)>0 means that the symmetric matrix (.) is positive definite. Also tr(.) will denote the trace of (.) and $\chi(\alpha)$ will denote the real part of (.). The normalizing constant c can be evaluated by using the following transformations. Let

$$\tilde{Y} = \tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}^{\frac{1}{2}} \Rightarrow d\tilde{Y} = |\tilde{A}|^{\frac{1}{2}}|\tilde{B}|^{p-\frac{1}{2}} d\tilde{X}$$

by using Theorem 1-18 of [3]. Let

$$\tilde{U} = \tilde{Y}\tilde{Y}' \Rightarrow d\tilde{Y} = \frac{\pi^{r\left(p-\frac{1}{2}\right)}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)}|\tilde{U}|^{\frac{r}{2}-p} d\tilde{U}$$

by using Theorem 2.16 of [3], where for example,

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p\left(\alpha - \frac{1}{2}\right)...\tilde{\Gamma}_p(\alpha-(p-1)), \alpha > p-1 \qquad ...(1.2)$$

taking α as real, and if complex the condition is $\chi(\alpha) > p-1$. Let

$$\tilde{V} = a(1-q)\tilde{U} \Rightarrow d\tilde{V} = [a-(1-q)]^{p^2} d\tilde{U}$$

by using Theorem 1.20 of [3], Then

$$1 = \int_{\tilde{X}} f(\tilde{X}) d\tilde{X} = \frac{c}{|\tilde{A}|^{\frac{r}{2}} |\tilde{B}|^{p-\frac{1}{2}}} \int_{\tilde{Y}} |\tilde{Y}\tilde{Y}'|^{\alpha} |I - a(1-q)\tilde{Y}\tilde{Y}'|^{\frac{\beta}{1-q}} d\tilde{Y}$$

$$= \frac{\pi^{\frac{rp}{2}}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{p-\frac{1}{2}}} \int_{\tilde{U}} |\tilde{U}|^{\alpha+\frac{r}{2}-p} |I - a(1-q)\tilde{U}|^{\frac{\beta}{1-q}} d\tilde{U} \qquad ...(1.3)$$

At this stage we can consider three possibilities: (i) q<1, (ii) q>1, (iii) q=1.

Let us consider these one by one.

Case (i) : q<1.

Then a(1-q) > 0 and then by making the transformation $\tilde{V} = a(1-q)\tilde{U}$, we have

$$c^{-1} = \frac{\pi^{r\left(p-\frac{1}{2}\right)}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{p-\frac{1}{2}}[a(1-q)]^{2p-1\left(\alpha+\frac{r}{2}\right)}} \int_{\tilde{V}} |\tilde{V}|^{\alpha+\frac{r}{2}-p} |I - \tilde{V}|^{\frac{\beta}{1-q}} d\tilde{V} \qquad ...(1.4)$$

Now, evaluating the integral in (1.4) by using a matrix- variate type-1 beta, see Section 5.1.4 of [3], we have

$$c^{-1} = \frac{\pi^{r\left(p-\frac{1}{2}\right)}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{p-\frac{1}{2}}[a(1-q)]^{2p-1\left(\alpha+\frac{r}{2}\right)}} \frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}_P\left(\frac{\beta}{1-q}+p\right)}{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p\right)} \qquad ...(1.5)$$

for $\alpha + \frac{r}{2} > p-1$ We will assume the parameters to be real for convenience.

Case (ii) : q > 1

In this case write 1-q= -(q-1) so that q-1 > 0. Then in (1.3)

$$|I - a(1-q)\tilde{U}|^{\frac{\beta}{1-q}} = |I + a(q-1)\tilde{U}|^{-\frac{\beta}{q-1}} \qquad ...(1.6)$$

and then make the transformation $\tilde{V} = a(q-1)\tilde{U}$. Then

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{p-\frac{1}{2}}[a(q-1)]^{2p-1\left[\alpha+\frac{r}{2}\right]}} \int_{\tilde{V}} |\tilde{V}|^{\alpha+\frac{r}{2}-p} |I+\tilde{V}|^{\frac{\beta}{q-1}} d\tilde{V}$$

Evaluating the integral by using a matrix-variate type-2 beta integral, see Section 5.1.4 of [3], we have the following :

$$c^{-1} = \frac{\pi^{\frac{rp}{2}}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{p-\frac{1}{2}}[a(q-1)]^{2p-1\left(\alpha+\frac{r}{2}\right)}} \frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}_P\left(\frac{\beta}{q-1}-\alpha-\frac{r}{2}\right)}{\tilde{\Gamma}_P\left(\frac{\beta}{q-1}\right)} \quad ...(1.7)$$

for $\alpha + \frac{r}{2} > p-1, \frac{\beta}{q-1} - \alpha - \frac{r}{2} > p-1$

Case (ii) : q = 1

Irrespective of whether q approaches 1 from the left or from the right it can be shown that the determinant containing q in (1.3) and (1.6) has the following form. which will be stated as a lemma:

Lemma 1.1:

$$\lim_{q\to 1} |I - a(1-q)\tilde{U}|^{\frac{\beta}{1-q}} = e^{-\alpha\beta\, tr(\tilde{U})}$$

This result can be seen by observing the following: For a symmetric positive definite matrix $\tilde{U}$ there exists a matrix $\tilde{Q}$ such that

$$\tilde{Q}\tilde{Q}' = I, \quad \tilde{Q}'\tilde{Q} = I, \quad \tilde{Q}'\tilde{U}\tilde{Q} = diag(\lambda_1,...\lambda_p), \lambda_j > 0, \; j = 1,..., p \quad ...(1.8)$$

where $diag(\lambda_1,...\lambda_p)$ denotes a diagonal matrix with the diagonal elements $\lambda_1,...\lambda_p$ Then

$$|I - a(1-q)\tilde{U}| = |I - a(1-q)\tilde{Q}\tilde{Q}'\tilde{U}\tilde{Q}\tilde{Q}'|$$

$$= |I - a(1-q)\tilde{Q}'\tilde{U}\tilde{Q}| = |I - a(1-q)diag(\lambda_1...\lambda_p)|$$

$$= \prod_{j=1}^{p} (1 - a(1-q)\lambda_j)$$

But

$$\lim_{q\to 1}\left(I-a(1-q)\lambda_j\right)^{\frac{\beta}{1-q}}=e^{-\alpha\beta\lambda_j}$$

Then

$$\lim_{q\to 1}\left|I-a(1-q)U\right|^{\frac{\beta}{1-q}}=e^{-\alpha\beta\left(\sum_{j=1}^{p}\lambda_j\right)}=e^{-\alpha\beta\, tr(\tilde{U})}$$

which establishes the result. Hence in case (iii)

$$c^{-1}=\frac{\pi^{r\left(p-\frac{1}{2}\right)}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{\frac{p}{2}}}\int_{\tilde{U}}|\tilde{U}|^{\alpha+\frac{r}{2}-p-e^{-\alpha\beta\, tr(\tilde{U})}}\, d\tilde{U}$$

$$=\frac{\pi^{r\left(p-\frac{1}{2}\right)}}{\tilde{\Gamma}_p\left(\frac{r}{2}\right)|\tilde{A}|^{\frac{r}{2}}|\tilde{B}|^{\frac{p}{2}}}\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}\right)}{(\alpha\beta)^{\left(\alpha+\frac{r}{2}\right)(2p-1)}}\quad ;\ \alpha+\frac{r}{2}>p-1 \qquad ...(1.9)$$

by using Section 5.1.1 of [3].

2. A General density:

For $\tilde{X},\tilde{A},\tilde{B},\alpha,\beta,q$ as defined in (1.1) let

$$f(\tilde{X})=c\left|\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^{\alpha}\left|I-a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^{\frac{\beta}{1-q}} \qquad ...(2.1)$$

for $q\neq 1$, and for $q=1$

$$=c\left|\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^{\alpha}e^{-\alpha\beta tr\left[\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right]} \qquad ...(2.2)$$

where c in (2.1) is given by (1.5) for q<1 and by (1.7) for q>1. From (1.9) we have the c in (2.2). In (2.1) a necessary condition to be met is that $I-a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}$ >0. Note that when q moves from $-\infty$ *to* 1, *that is*, $-\infty<q<1$ then (2.1) maintains a matrix-variate type-1beta form and when q becomes greater than 1 then the type-1 beta form switches to a type-2 beta form. That is, to the left of 1 for q a type-1 beta form is available and to the right of 1 for q a type-2 beta form is available. Both these type-1 and type-2 beta forms go to a matrix-variate gamma form at q=1.

Thus the pathway for q describes a wide range of statistical densities covering type-1 and type-2 beta forms and gamma forms. It may also be noted from (1.1) that one need not go for the symmetric square roots $\tilde{A}^{\frac{1}{2}}$ and $\tilde{B}^{\frac{1}{2}}$ of $\tilde{A}$ and $\tilde{B}$, one needs to obtain only a representation $\tilde{A}=\tilde{A}_1\tilde{A}_1'$ and $\tilde{B}=\tilde{B}_1\tilde{B}_1'$. Then one $\tilde{A}^{\frac{1}{2}}$ could be replaced by $\tilde{A}_1'$ and one $\tilde{B}^{\frac{1}{2}}$ by $\tilde{B}_1'$.

2.1 Arbitrary moments:

Arbitrary h^{th} moment for the determinant $\left|\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$ or that for $\left|\tilde{X}\tilde{B}\tilde{X}'\right|$ can be obtained from c^{-1} in (1.5), (1.7), (1.9) for the cases q<1, q>1, q=1 respectively, by changing α to $\alpha+h$ and then taking the ratio of the normalizing constants. Thus we have the following, where E denotes the expected value.

Theorem 2.1

$$E\left|\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^h=\frac{1}{[a(1-q)]^{(2p-1)h}}\frac{\tilde{\Gamma}_p\left(\alpha+h+\frac{r}{2}\right)}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)}\frac{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}{\tilde{\Gamma}_p\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}$$

$$\text{for } q<1, \alpha+h+\frac{r}{2}>p-1 \qquad \ldots(2.3)$$

$$=\frac{1}{[a(q-1)]^{(2p-1)h}}\frac{\tilde{\Gamma}_p\left(\alpha+h+\frac{r}{2}\right)}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)}\frac{\tilde{\Gamma}_p\left(\frac{\beta}{q-1}-\alpha-h-\frac{r}{2}\right)}{\tilde{\Gamma}_p\left(\frac{\beta}{q-1}-\alpha-\frac{r}{2}\right)} \qquad \ldots(2.4)$$

$$\text{for } q>1, \frac{\beta}{q-1}-\alpha-h-\frac{r}{2}>p-1, \alpha+h+\frac{r}{2}>p-1$$

$$=\frac{1}{[a\beta]^{(2p-1)h}}\frac{\tilde{\Gamma}_p\left(\alpha+h+\frac{r}{2}\right)}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)} \text{ for } q=1, \alpha+h+\frac{r}{2}>p-1 \qquad \ldots(2.5)$$

One may wonder whether (2.3) and (2.4) go to (2.5) when $q\to 1$ from the left and right respectively. This can be seen from an asymptotic expansion for gamma functions or from Stirling's approximation. These will be stated as lemmas.

Lemma 2.1: For $|z| \to \infty$ and a bounded quantity,

$$\tilde{\Gamma}(z+a) \approx \sqrt{2\pi}\, z^{z+a-\frac{1}{2}} e^{-z} \qquad \ldots(2.6)$$

Where $\approx$ means "approximately equal to".

Then by applying lemma 2.1 and writing $\tilde{\Gamma}_p(\bullet)$ in explicit forms one has the following results.

Lemma 2.2: $$\lim_{q\to1}\left\{\frac{1}{[a(1-q)]^{(2p-1)h}}\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}{\tilde{\Gamma}_P\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}\right\}=\frac{1}{[\alpha\beta]^{(2p-1)h}} \qquad \ldots(2.7)$$

This can be seen by observing the following:

$$\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}{\tilde{\Gamma}_P\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}=\prod_{j=1}^{p}\left[\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)}{\tilde{\Gamma}_P\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)}\right]$$

When q goes to 1 from the left $\frac{\beta}{1-q}\to\infty$ then for example.

$$\prod_{j=1}^{p}\tilde{\Gamma}\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)e^{-\frac{\beta}{1-q}}=\prod_{j=1}^{p}\sqrt{2\pi}\left(\frac{\beta}{1-q}\right)^{\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}-\frac{1}{2}} e^{-\frac{p\beta}{1-q}}$$

$$=\left(\sqrt{2\pi}\right)^{2p-1}\left(\frac{\beta}{1-q}\right)^{2p-1\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}\right)+\frac{(2p-1)p}{2}}$$

Hence,

$$\frac{1}{[a(1-q)]^{(2p-1)h}}\prod_{j=1}^{p}\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)}{\tilde{\Gamma}_P\left(\alpha+h+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)}=\frac{1}{[\alpha\beta]^{(2p-1)h}}$$

This establishes that (2.3) goes to (2.5) when $q\to1$ from the left. In a similar way one can see that (2.4) also goes to (2.5). Thus q is a pathway from moments in (2.3) and (2.4) to go to the moments in (2.5).

One can make some interesting observations from (2.3)-(2.5). From (2.3) we have,

$$E\left|a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|^h = \prod_{j=1}^{p}\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+h-\frac{j-1}{2}\right)}{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}-\frac{j-1}{2}\right)}\frac{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p-\frac{j-1}{2}\right)}{\tilde{\Gamma}_P\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p+h--\frac{j-1}{2}\right)}$$

$$= \prod_{j=1}^{p}E\left(x_j^h\right) \qquad \ldots(2.8)$$

where x_j is a real scalar type-1 beta random variable with the parameters

$$\left(\alpha+\frac{r}{2}-\frac{j-1}{2},\frac{\beta}{1-q}+p\right),\ j=1,\ldots,p.$$

Thus, structurally, $\left|a(1-q)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$, for q<1, is a product p of statistically independently distributed real type-1 beta random variables with the parameters as mentioned above. Similarly for q>1, $\left|a(q-1)\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$ is a product of p statistically independently distributed type-2 real scalar beta random variables, and from (2.5), $\left|\alpha\beta\tilde{A}^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$ is a product of p independently distributed gamma random variables. These products of independent scalar type-1 beta and type-2 beta random variables go to a product of independent scalar gamma random variables when $q\to 1$. Thus, through q a pathway is achieved to go to product of independent gamma variables from products of independent type-1 beta and type-2 beta variables.

Theorem 2.2: When p=1, r > p then the density of $u=A^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}=(x_1,\ldots x_r)B[x_1\ldots x_r]^T$ is given by

$$g(u)=c_1u^{\alpha+\frac{r}{2}-1}[1-a(1-q)u]^{\frac{\beta}{1-q}} \qquad \ldots(2.9)$$

with 1-a(1-q)u>0, where, for q< 1

$$c_1=\frac{[a(1-q)]^{\alpha+\frac{r}{2}}\tilde{\Gamma}\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+1\right)}{\tilde{\Gamma}\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}\left(\frac{\beta}{1-q}+1\right)},\ \alpha+\frac{r}{2}>0 \qquad \ldots(2.10)$$

for q > 1

$$c_1 = \frac{[a(q-1)]^{\alpha+\frac{r}{2}}\tilde{\Gamma}\left(\frac{\beta}{q-1}\right)}{\tilde{\Gamma}\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}\left(\frac{\beta}{q-1}-\alpha-\frac{r}{2}\right)}, \alpha+\frac{r}{2}>0, \frac{\beta}{q-1}-\alpha-\frac{r}{2}>0 \qquad \ldots(2.11)$$

and for q = 1

$$c_1 = \frac{(\alpha\beta)^{\alpha+\frac{r}{2}}}{\tilde{\Gamma}\left(\alpha+\frac{r}{2}\right)}, \ \alpha+\frac{r}{2}>0. \qquad \ldots(2.12)$$

Distributions of quadratic forms in real Gaussian random variables are discussed in [6] and the distributions of generalized quadratic forms with Gaussian vector random variables are considered in [7]. But if the $p\times r, r\geq p$ random matrix $\tilde{X}$ has a matrix-variate distribution as in (1.1), which covers rectangular matrix-variate type-1 beta, type-2 beta, gamma type and Gaussian type distributions, then the density of the generalized quadratic form follows trivially from (1.1). This will be gives as the next theorem.

Theorem 2.3: When the $p\times r, r\geq p$ random matrix $\tilde{X}$ has the matrix variate distribution as given in (1.1) then the generalized quadratic form $\tilde{Y} = \tilde{A}^{\frac{1}{2}}\tilde{B}\tilde{X}\tilde{B}'\tilde{A}^{\frac{1}{2}}$ has the following density, denoted by

$$f_1(\tilde{Y}) = c_2|\tilde{Y}|^{\alpha+\frac{r}{2}-p}|I-a(1-q)\tilde{Y}|^{\frac{\beta}{1-q}} \qquad \ldots(2.13)$$

Where, for q > 1

$$c_2 = \frac{[a(1-q)]^{(2p-1)\left(\alpha+\frac{r}{2}\right)}\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}+\frac{\beta}{1-q}+p\right)}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}_p\left(\frac{\beta}{1-q}+p\right)}, \alpha+\frac{r}{2}>p-1 \qquad \ldots(2.14)$$

for q > 1

$$c_2 = \frac{[a(q-1)]^{(2p-1)\left(\alpha+\frac{r}{2}\right)}\tilde{\Gamma}_p\left(\frac{\beta}{q-1}\right)}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)\tilde{\Gamma}_p\left(\frac{\beta}{q-1}-\alpha-\frac{r}{2}\right)}; \ \alpha+\frac{r}{2}>p-1, \ \frac{\beta}{q-1}-\alpha-\frac{r}{2}>p-1 \qquad \ldots(2.15)$$

for q = 1

$$c_2 = \frac{(\alpha\beta)^{(2p-1)\left(\alpha+\frac{r}{2}\right)}}{\tilde{\Gamma}_p\left(\alpha+\frac{r}{2}\right)}, \ \alpha+\frac{r}{2} > p-1 \qquad \ldots(2.16)$$

3. Connection to geometrical probability problems:

While considering the distributional aspects of the volume content of a r-parallelotope generated by the convex hull of linearly independent random points in Euclidean n-space many authors had considered the problem when the points are isotropic and are distributed according to a beta type-1, type-2 and Gaussian situations. The distributions of the random points that they considered were particular cases of (2.9) with $\tilde{B} = I$. More general situations in this category of problems are considered in [4]. Since the determinant of the type $\left|A^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$, appearing in (1.1), can be considered to be volume of an appropriately defined parallelotope a more general model in this category of problems is available form (1.1). Note that the p x r matrix $\tilde{X}$ of full rank can also be looked upon as p linearly independent points in a r-dimensional Euclidean space. Then $\left|\tilde{X}\tilde{X}'\right|$ is the square of the volume of the parallelotope generated by the convex hull of these p points in r-space, $r \geq p$. *Hence* $\left|a(1-q)A^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$ is the square of the volume of the parallelotope generated by p points in a transformed space. Also from (2.3)-(2.5) it is seen that $\left|A^{\frac{1}{2}}\tilde{X}\tilde{B}\tilde{X}'\tilde{A}^{\frac{1}{2}}\right|$ is structurally a product of p independent type-1 beta type-2 beta and gamma random variables corresponding to q<1, q>1 and q=1 respectively.

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