Solving Berger's Equation Arising in Fingering Phenomenon by Laplace Transform Method

1 Mansi K. Desai 2 Dr.Shailesh S. Patel

1 Ph.D. Section / Gujarat Technological University, Chandkheda, Ahmedabad

2 Applied Science & Humanities Department /GIDC Degree Engineering college, Navsari

ABSTRACT

The present paper discusses analytically the phenomenon of Fingering in double phase flow through homogenous porous media by using Laplace Transform Method. The basic assumptions underlying in the present investigation is that the oil and water form two immiscible liquid phases and the latter represents preferentially wetting phase. The Saturation of injected water is calculated by Laplace Transform method for Berger's Equation of Fingering phenomenon under assumption that Saturation is decomposed in to saturation of different levels. The obtained results as compared with previous works are highly accurate. Also Laplace Transform Method provides continuous solution in contrast to finite difference method, which only provides discrete approximations. Graphical illustration has been done by mat lab.

KEY WORDS

Fingering Phenomenon, Berger's Equation, Laplace Transform method.

INTRODUCTION

The present paper discusses the classical solution of burger's equation which arises into the fingering phenomenon in double phase flow through homogenous porous media. When a fluid flowing through a porous media is displaced by another fluid of lesser viscosity then, instead of regular displacement of the whole front, protuberance takes place which shoot through the porous medium at a relatively very high speed. The occurrence of protuberance is called as the instability phenomenon or fingering. For mathematical models it is assumed that there is a uniform water injection into an oil saturated porous medium. The injected water shoots through the oil formation and gives rise to protuberance (fingers). Finally by the comment by Parkes[1], it has been proved that the solution of perturb burger's equation given by Demiray[3] is solution of burger's equation by identification of the error and some over looked the fact by Demiray[3].It is also suggested by Parkes[1] that as $\gamma_3 = 0$ which, as mentioned previously, leads to a solution to the burger's equation and not, as required, to a solution to the perturb burger's equation. The finally, it is solution of burger's equation which represent saturation of the i^{th} fluid at level, x, for any time $t \geq 0$.

Mathematical Formulation of the Problem and Its Solution

The seepage velocity of water and oil are given by Darcy's Law as

$$V_{iw} = -\frac{K_{iw}}{\mu_{iw}} K \left(\frac{\partial P_{iw}}{\partial x} \right) \tag{1}$$

$$V_{no} = -\frac{K_{no}}{\mu_{no}} K \left(\frac{\partial P_{no}}{\partial x} \right) \tag{2}$$

Where,

K = Permeability of homogenous porous medium

μ_{no} = The constant kinematic viscosities of oil

μ_{iw} = The constant kinematic viscosities of water

P_{no} = Pressures of oil

P_{iw} = Pressures of water

K_{no} = Relative permeability of oil

K_{iw} = Relative permeability of water

The equation of continuity of two phases densities are regarded as constant, is given as

$$P \left(\frac{\partial S_{iw}}{\partial t} \right) + \left(\frac{\partial v_{iw}}{\partial x} \right) = 0 \tag{3}$$

$$P \left(\frac{\partial S_{no}}{\partial t} \right) + \left(\frac{\partial v_{no}}{\partial x} \right) = 0 \tag{4}$$

Where P is the porosity of the medium.

From the definition of phase saturation, it is evident that

$$S_{iw} + S_{no} = 1 \tag{5}$$

The capillary pressure p_n , defined as discontinuity of the flowing phase across their common interface, is a function of the phase saturation. It may be written as

$$p_{no} = p_c - p_{iw} \tag{6}$$

The equation of motion for saturation is obtained by sub-stituting the values of V_{iw} and V_{no}

from equations (1) and (2) to the equations (3) and (4) respectively,

$$P\left(\frac{\partial s_{iw}}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_{iw}}{\delta_{iw}}\right) K \left(\frac{\partial p_{iw}}{\partial x}\right)\right] \tag{7}$$

$$P\left(\frac{\partial s_{no}}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_{no}}{\delta_{no}}\right) K \left(\frac{\partial p_{no}}{\partial x}\right)\right] \tag{8}$$

Eliminating $\frac{\partial p_{iw}}{\partial x}$ form equation (6) & (7) we have,

$$P\left(\frac{\partial s_{iw}}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_{iw}}{\delta_{iw}}\right) K \left\{\left(\frac{\partial p_{no}}{\partial x}\right) - \left(\frac{\partial p_c}{\partial x}\right)\right\}\right] \tag{9}$$

By combining equations (8) and (9), and using equation (5) , we get

$$\frac{\partial}{\partial x}\left[\left\{\left(\frac{k_{iw}}{\delta_{iw}}\right) K + \left(\frac{\partial p_{no}}{\partial x}\right) K\right\} \left(\frac{\partial p_{no}}{\partial x}\right) - \left(\frac{k_{iw}}{\delta_{iw}}\right) K \left(\frac{\partial p_c}{\partial x}\right)\right] = 0 \tag{10}$$

Integrating equation (1.10) with respect to $'x'$, we get,

$$\left[\left(\frac{k_{iw}}{\delta_{iw}}\right) K + \left(\frac{k_{no}}{\delta_{no}}\right) K\right]\left(\frac{\partial p_{no}}{\partial x}\right) - \left(\frac{k_{iw}}{\delta_{iw}}\right) K \left(\frac{\partial p_c}{\partial x}\right) = -V \tag{11}$$

Where V is a constant of integration, which can be evaluated later on. By simplify (11), we get

$$\frac{\partial p_{no}}{\partial x} = -\frac{V}{K\left[\left(\frac{k_{no}}{\delta_{no}}\right)+\left(\frac{k_{iw}}{\delta_{iw}}\right)\right]} + \frac{\left(\frac{\partial p_C}{\partial x}\right)}{1+\left(\frac{k_{no}}{k_{iw}}\right)\left(\frac{\delta_{iw}}{\delta_{no}}\right)} \tag{12}$$

From equation (1.12), the following is obtained

$$P\,\frac{\partial s_{iw}}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\left(\frac{k_{no}}{\delta_{no}}\right)\left(\frac{\partial p_C}{\partial x}\right)}{1+\left[\left(\frac{k_{no}}{k_{iw}}\right)+\left(\frac{\delta_{iw}}{\delta_{no}}\right)\right]} + \frac{V}{1+\left(\frac{k_{no}}{k_{iw}}\right)\left(\frac{\delta_{iw}}{\delta_{no}}\right)}\right] = 0 \tag{13}$$

The value of the pressure of oil (P_{no}) can be written as,

$$p_{no} = \bar{P} + \frac{1}{2}\,p_c\,, \quad \bar{P} = \frac{p_{no}+p_{iw}}{2} \tag{14}$$

Where $\bar{P}$ is the constant mean pressure.

On differentiating the above equation with respect to x, the following equation is obtained,

$$\frac{\partial p_{no}}{\partial x} = \frac{1}{2}\frac{\partial p_c}{\partial x} \tag{15}$$

On sub stituting the value of $\frac{\partial p_{no}}{\partial x}$ in equation (11) we can obtained,

$$V = \left[\frac{K}{2}\left(\frac{k_{iw}}{\delta_{iw}} - \frac{k_{no}}{\delta_{no}}\right)\frac{\partial p_c}{\partial x}\right] \tag{16}$$

On substuting the value of V from equation (16) and (13), we can obtained,

$$P\,\frac{\partial S_{iw}}{\partial t} + \frac{1}{2}\,\frac{\partial}{\partial x}\left[K\left(\frac{k_{iw}}{\delta_{iw}}\,\frac{dp_c}{dS_{iw}}\,\frac{\partial S_{iw}}{\partial x}\right)\right] = 0 \tag{17}$$

$$P\,\frac{\partial S_{iw}}{\partial t} + \frac{1}{2}\,\frac{\partial}{\partial x}\left[\left(D(S_{iw})\frac{\partial S_{iw}}{\partial x}\right)\right] = 0 \quad \text{where } D(S_{iw}) = K\left(\frac{k_{iw}}{\delta_{iw}}\,\frac{dp_c}{dS_i}\right)$$

Now, from equation (17) we get,

$$\frac{\partial S_{iw}}{\partial t} = a^2\,\frac{\partial^2 S_{iw}}{\partial x^2} \qquad \text{where } a^2 = -\frac{D(S_{iw})}{2P} \tag{18}$$

Which is the burger's equation arises into the fingering phenomenon in double phase flow through porous media.

Now taking $S_{iw} = S$ in equation (18), equation (18) becomes,

$$\frac{\partial S}{\partial t} = a^2\,\frac{\partial^2 S}{\partial x^2} \qquad \text{where } a^2 = -\frac{D(\hat{S})}{2P} \tag{19}$$

Now, solving the burger's equation with appropriate boundary conditions,

$$S_t(0,t) = 0 \quad \text{and} \quad S(l,t) + S_t(l,t) = 1 \tag{20}$$

Taking initial condition,

$$S_x(x,0) = 0 \tag{21}$$

Now taking Laplace transforms both the sides of equation (1), we get,

$$s\hat{S} = a^2\hat{S}_{xx} \quad \text{and} \quad \hat{S}(0,s) = 0 \;\&\; \hat{S}(l,s) + \hat{S}_x(l,s) = \frac{1}{s}$$

$$\hat{S}_{xx} - \frac{s}{a^2}\hat{S} = 0 \; ; \; \hat{S}(0,s) = 0 \;\&\; \hat{S}(l,s) + \hat{S}_x(l,s) = \frac{1}{s}$$

The Solution that satisfies the left boundary condition is,

$$\hat{S} = c\sinh(\frac{\sqrt{s}x}{a})$$

We apply the right boundary condition to determine the constant.

$$\hat{S} = \frac{\sinh(\frac{\sqrt{s}x}{a})}{s(\sinh\left(\frac{\sqrt{s}l}{a}\right) + \frac{\sqrt{s}}{a}\cosh(\frac{\sqrt{s}l}{a}))}$$

We expand this in a series of simpler function of s,

$$\hat{S} = \frac{2\sinh(\frac{\sqrt{s}x}{a})}{s\left[\exp\left(\frac{\sqrt{s}l}{a}\right) - \exp\left(-\frac{\sqrt{s}l}{a}\right) + \frac{\sqrt{s}}{a}\left(exp\left(\frac{\sqrt{s}l}{a}\right) + \exp\left(-\frac{\sqrt{s}l}{a}\right)\right)\right]}$$

$$\hat{S} = \frac{2\sinh(\frac{\sqrt{s}x}{a})}{s\exp(\frac{\sqrt{s}x}{a})} \quad \frac{1}{1+\frac{\sqrt{s}}{a} - \left(1-\frac{\sqrt{s}}{a}\right)\exp\left(-\frac{2\sqrt{s}l}{a}\right)}$$

$$\hat{S} = \frac{exp\left(\frac{\sqrt{s}x}{a}\right) - \exp\left(\frac{\sqrt{s}x}{a}\right)}{s\left(1+\frac{\sqrt{s}}{a}\right)\exp\left(\frac{\sqrt{s}l}{a}\right)} \; \frac{1}{1 - \left(\frac{1-\frac{\sqrt{s}}{a}}{1+\frac{\sqrt{s}}{a}}\right)\exp(-\frac{2\sqrt{s}l}{a})}$$

$$\hat{S} = \frac{\exp\left(\frac{\sqrt{s}(x-1)}{a}\right) - \exp\left(\frac{\sqrt{s}(x-1)}{a}\right)}{s\left(1+\frac{\sqrt{s}}{a}\right)} \; \sum_{n=0}^{\infty}\left(\frac{1-\frac{\sqrt{s}}{a}}{1+\frac{\sqrt{s}}{a}}\right)^n \; \exp\left(-\frac{2\frac{\sqrt{s}}{n}}{a}\right)$$

$$\hat{S} = \frac{1}{s}\left(\sum_{n=0}^{\infty}\frac{\left(1-\frac{\sqrt{s}}{a}\right)^{n}}{\left(1+\frac{\sqrt{s}}{a}\right)^{n+1}}\exp\left(-\frac{\sqrt{s}((2n-1)l-x)}{a}\right) - \sum_{n=0}^{\infty}\frac{\left(1-\frac{\sqrt{s}}{a}\right)^{n}}{\left(1+\frac{\sqrt{s}}{a}\right)^{n+1}}\exp\left(-\frac{\sqrt{s}((2n-1)l+x)}{a}\right)\right)$$

By expanding, $\frac{\left(1-\frac{\sqrt{s}}{a}\right)^{n}}{\left(1+\frac{\sqrt{s}}{a}\right)^{n+1}}$

In binomial series all the terms would be of the form,

$$s^{-m/2-3/2}\exp\left(-\frac{\sqrt{s}((2n-1)l\mp x)}{a}\right)$$

Taking the first term in each series yields

$$\hat{S} = \frac{a}{s^{3/2}}\left(\exp\left(-\frac{\sqrt{s}(l-x)}{a}\right) - \left(-\frac{\sqrt{s}(l+x)}{a}\right)\right), as\ s \to \infty.$$

We take the inverse Laplace transform to obtain an approximate of the solution for $t \ll 1$.

$$\hat{S}(x,t) \sim 2a^2\sqrt{\pi t}\left(\frac{\exp\left(-\frac{(l-x)^2}{4a^2t}\right)}{l-x} - \frac{\exp\left(-\frac{(l+x)^2}{4a^2t}\right)}{l+x}\right) - \pi\left(erfc\left(\frac{(l-x)}{2a\sqrt{t}}\right) - erfc\left(\frac{(l+x)}{2a\sqrt{t}}\right)\right), \quad (22)$$

$$for\ t \ll 1.$$

CONCLUSIONS

Here the solution of the fingering phenomenon has obtained given by Demiray [3] is not solution of perturb burger's equation but it is accepted as solution of burger's equation (20) with appropriate boundary conditions. The solution satisfies both boundary conditions. The solution is in terms of hyperbolic as well as exponential terms. The solution procedure is simple, but the calculation of polynomials is complex. To overcome this shortcoming, Laplace transform method is used.

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