Flexural Response of Simply Supported Beam Subjected to Cosine Load by TSDT

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ABSTRACT: A trigonometric shear deformation theory for flexure of thick beams, taking into account transverse shear deformation effects, is developed. Governing differential equations and boundary conditions are obtained by using the principle of virtual work. The thick isotropic beams are considered for the numerical studies to demonstrate the efficiency of the theory. The simply supported isotropic beams subjected to cosine loads are examined using the present theory. Results obtained are discussed critically with those of other theories. This paper proposes a trigonometri shear deformation theory for thick isotropic simply supported beam. The displacement field of the present theory was based on a two variable, in which the transverse displacement is partitioned into the bending and shear parts. The proposed theories exactly satisfy the transverse stress boundary conditions on the bottom and top surfaces of the beam which were true in earlier shear deformation theories also. Beam governing equations and boundary conditions are derived by employing the principle of virtual work. The displacement and stresses of simply supported beam under cosine load are calculated to verify the accuracy and efficiency of the present theory. Numerical results indicate that the obtained predictions are comparable with other shear deformation theories.

Keywords: Thick beam, trigonometric shear deformation, principle of virtual work, equilibrium equations, displacement, stress.

1. INTRODUCTION

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for slender beams and is not suitable for thick or deep beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation, it underestimates deflections in case of thick beams where shear deformation effects are significant. Bresse [1], Rayleigh [2] and Timoshenko [3] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. Cowper [4] has given refined expression for the shear correction factor for different cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [5] with a plane stress exact elasticity solution. To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beam. Levinson [6], Bickford [7], Rehfield and Murty [8], Krishna Murty [9], Baluch, Azad and Khidir [10], Bhimaraddi and Chandrashekhara [11] presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor. Irretier [12] studied the refined dynamical effects in linear, homogenous beam according to theories, which exceed the limits of the Euler- Bernoulli beam theory. These effects are rotary inertia, shear deformation, rotary inertia and shear deformation, axial pre-stress, twist and coupling between bending and torsion. Kant and Gupta [13], Heyliger and Reddy [14] presented finite element models based on higher order shear deformation uniform rectangular beams. However, these displacement based finite element models are not free from phenomenon of shear locking (Averill and Reddy [15], Reddy [16]). There is another class of refined theories, which includes trigonometric functions to represent the shear deformation effects through the thickness. Vlasov and Leont'ev [17], Stein [18] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. A study of literature by Ghugal and Shimpi [19] indicates that the research work dealing with flexural analysis of thick beams using refined trigonometric and hyperbolic shear deformation theories is very scarce and is still in infancy. In this paper development of theory and its application to thick simply supported beam is presented.

2. DEVELOPMENT OF THEORY

The beam under consideration as shown in Fig. 1 occupies in *x-y-z* Cartesian coordinate system the region: Where *x*, *y*, *z* are Cartesian coordinates, *L* and *b* are the length and width of beam in the *x* and *y* directions respectively, and *h* is the thickness of the beam in the *z*-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

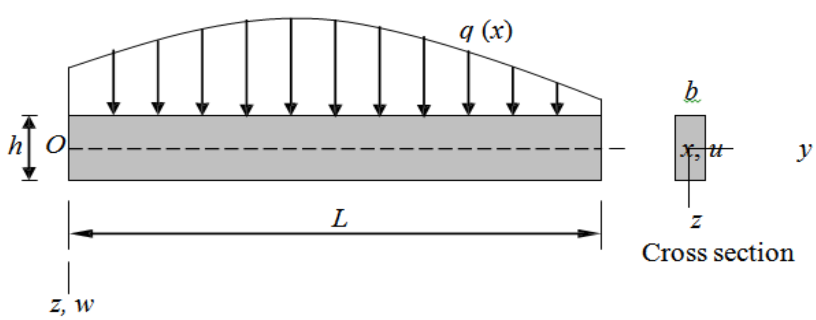

Fig.1 Beam under bending in *x-z* plane

A. Assumptions made in the theoretical formulation:

1. The axial displacement *(u)* consists of two parts:
   a) Displacement given by elementary theory of bending.
   b) Displacement due to shear deformation, which is assumed to be hyperbolic in nature with respect to thickness coordinate, such that maximum shear stress occurs at neutral axis as predicted by the elementary theory of bending of beam.
2. The axial displacement (*u)* is such that the resultant of in-plane stress *(σ_x)* acting over the cross-section should result in only bending moment and should not in force in *x* direction.
3. The transverse displacement *(w)* in *z* direction is assumed to be function of *x* coordinate.
4. The displacements are small as compared to beam thickness.
5. The body forces are ignored in the analysis. (The body forces can be effectively taken into account by adding them to the external forces.)
6. One dimensional constitutive law are used.

The beam is subjected to lateral load only.

B. The Displacement Field:

Based on the above mentioned assumptions, the displacement field of the present beam theory can be expressed as follows.

$$u(x,z) = -z\frac{dw}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x) \qquad (01)$$

$$w(x,z) = w(x) \qquad (02)$$

Where,

u = Axial displacement in *x* direction which is function of *x* and *z.*

w = Transverse displacement in *z* direction which is function of *x.*

ϕ = Rotation of cross section of beam at neutral axis which is function of *x*

Normal Strain:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \qquad (03)$$

Shear Strain:

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \cos\frac{\pi z}{h}\phi \qquad (04)$$

Stress-strain relationships:

One dimensional Hooke's law is applied for isotropic material, stress *(σ_x)* is related to strain *(ϵ_x)* and shear stress is related to shear strain by the following constitutive relations:

$$\sigma_X = E\varepsilon_x = -Ez\frac{d^2 w}{dx^2} + \frac{Eh}{\pi}\sin\frac{\pi z}{h}\frac{d\phi}{dx} \qquad (05)$$

$$\tau_{zx} = G\gamma_{zx} = G\cos\frac{\pi z}{h}\phi \qquad (06)$$

where, E and G are the elastic constants of the beam material.

B. Governing Equations and Boundary Conditions

Governing differential equations and boundary conditions are obtained from Principle of virtual work. Using equations for stresses, strains and principle of virtual work, variationally consistent differential equations for beam under consideration are obtained. The principle of virtual work when applied to beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x \delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dxdz - \int_{X=0}^{X=L} q(x)\delta w dx = 0 \quad (07)$$

Where, δ = variational operator,

Employing Greens theorem in above equation successively, we obtained the coupled Euler-Langrange equations, which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4 w}{dx^4} - \frac{24}{\pi^3}EI\frac{d^3\phi}{dx^3} = q(x) \quad (08)$$

$$\frac{24}{\pi^3}EI\frac{d^3 w}{dx^3} - \frac{6}{\pi^2}EI\frac{d^2\phi}{dx^2} + \frac{GA}{2}\phi = 0 \quad (09)$$

The associated variationally consistent boundary conditions obtained at the ends $x = 0$ and $x = L$ is of following form:

Either $V_x = EI\frac{d^3 w}{dx^3} - \frac{24}{\pi^3}EI\frac{d^2\phi}{dx^2} = 0$ or or w is Prescribed (10)

Either $M_x = EI\frac{d^2 w}{dx^2} - \frac{24}{\pi^3}EI\frac{d\phi}{dx} = 0$ or or $\frac{dw}{dx}$ is Prescribed (11)

Either $M_s = EI\frac{24}{\pi^3}\frac{d^2 w}{dx^2} - \frac{6}{\pi^2}EI\frac{d\phi}{dx} = 0$ or or ϕ is Prescribed (12)

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

C. The General Solution of Governing Equilibrium Equations of the Beam

The general solution for transverse displacement *w(x)* and warping function*(x)* is obtained using (8) and (9) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging (8) we obtain the following expression.

$$\frac{d^3 w}{dx^3} = \frac{24}{\pi^3}\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI} \quad (13)$$

w

here $Q(x)$ is the generalized shear force for beam and it is given by $Q(x) = \int^x q dx + C_1$

Now (9) is rearranged in the following form:

$$\frac{d^3w}{dx^3} = \frac{\pi}{4}\frac{d^2\phi}{dx^2} - \beta\phi \qquad (14)$$

A single equation in terms of ϕ is now obtained using Eqns. (13) and (14) as:

$$\frac{d^2\phi}{dx^2} - \lambda^2\phi = \frac{Q(x)}{\alpha EI} \qquad (15)$$

where the constants α , β and λ appeared in Eqns. (13) and (14) are as follows:

$$\alpha = \left(\frac{\pi}{4} - \frac{24}{\pi^3}\right), \beta = \left(\frac{\pi^3}{48}\frac{GA}{EI}\right) \qquad (16)$$

$$\lambda^2 = \frac{\beta}{\alpha} \qquad (17)$$

$$\phi(x) = C_2 \cosh \lambda x + C_3 \sinh \lambda x - \frac{Q(x)}{\beta EI} \qquad (18)$$

The general solution for transverse displacement $w(x)$ can be obtained by substituting the value of ϕ *(x)* in Eqn. (15) and then integrating it thrice with respect to *x.* The solution is:

$$EIw(x) = \iiint q\,dxdxdx + \frac{C_1x^3}{6} + \left(\frac{\pi}{4}\lambda^2 - \beta\right)\frac{EI}{\lambda^3}\left(C_2 \sinh \lambda x + C_3 \cosh \lambda x\right) + C_4\frac{x^2}{2} + C_5x + C_6 \qquad (19)$$

where C1 , C2 , C3 , C4 , C5 and C6 are the arbitrary constants of integration and can be obtained by imposing natural (forced) and /or geometric or kinematical boundary / end conditions of beam.

3. ILLUSTARTIVE EXAMPLE

In order to prove the efficacy of the present theory, numerical example is considered. For the static flexural analysis, a uniform beam of rectangular cross section, having span length '*L',* width '*b'* and thickness '*h'* of homogeneous, elastic and isotropic material is considered. The following material properties for beam are used.

E = 210 GPa, μ = 0.3 and ρ = 7800 Kg/m^3

where E is the Young's modulus, ρ is the density, and μ is the Poisson's ratio of beam material. The kinematic and static (forced) boundary conditions associated with simply supported beam bending problems as follows:

Simply Supported Beam with Cosine Load,

The simply supported beam is having its origin at left support and is simply supported at $x=0$ and $x=L$. the beam is subjected to Cosine Load, $q_0(x) = q_0 \cos\frac{\pi x}{L}$ on surface $z = +h/2$ in the region $(0 \le x \le L/2)$ acting in the downward z direction and in the region $(L/2 \le x \le L)$ it acts in upward direction on surface $z = -h/2$ with maximum intensity of load q_0.

Boundary conditions associated with this problem are as follows:

Simple Supports: $EI\dfrac{d^2w}{dx^2} = EI\dfrac{d\phi}{dx} = w = 0$ at $x=0$

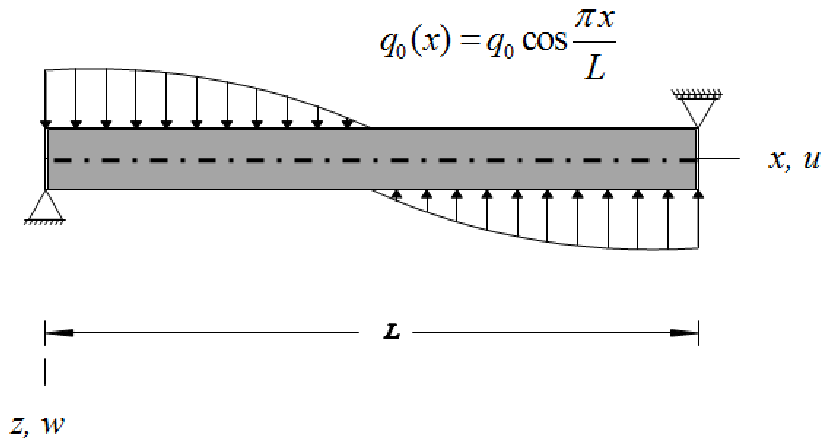

Fig. 2: Simply supported beam with cosine load

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

$$w(x) = \frac{q_0 L^4}{120EI}\left\{\begin{aligned} &\frac{120}{\pi^2}\left[\frac{1}{\pi^2}\left(\cos\frac{\pi x}{L} - 1 + 2\frac{x}{L}\right) + \frac{1}{2}\frac{x^2}{L^2} - \frac{1}{3}\frac{x^3}{L^3} - \frac{1}{6}\frac{x}{L}\right] \\ &+ \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}\left(-\frac{1}{2}\frac{x^2}{L^2} + \frac{1}{2}\frac{x}{L}\right) \\ &+ \frac{120}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{\pi^2}\left(\cos\frac{\pi x}{L} - 1 + \frac{x}{L}\right) + \frac{1}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x}{L}\right) \end{aligned}\right\} \tag{20}$$

$$\phi(\mathrm{x}) = \frac{q_0 L}{\beta EI}\left(\frac{2}{\pi^2} - \frac{1}{\pi}\sin\frac{\pi x}{L} + \frac{\sinh\lambda x - \cosh\lambda x}{\lambda L}\right) \tag{21}$$

The axial displacement and stresses obtained based on above solutions are as follows:

$$u(x) = \frac{q_0 h}{Eb}\left\{\begin{aligned} &-\frac{1}{10}\frac{z}{h}\frac{L^3}{h^3}\left[\begin{aligned} &\frac{120}{\pi^2}\left[\frac{1}{\pi^2}\left(-\pi\sin\frac{\pi x}{L} + 2\right) + \frac{x}{L} - \frac{x^2}{L^2} - \frac{1}{6}\right] \\ &+ \frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}\left(-\frac{x}{L} + \frac{1}{2}\right) \\ &+ \frac{120}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{\pi^2}\left(-\pi\sin\frac{\pi x}{L} + 2\right) + \frac{x}{L} - \frac{1}{2}\right) \end{aligned}\right] \\ &+ \frac{48}{\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\frac{L}{h}\left(\frac{2}{\pi^2} - \frac{1}{\pi}\sin\frac{\pi x}{L} + \frac{\sinh\lambda x - \cosh\lambda x}{\lambda L}\right) \end{aligned}\right\} \tag{22}$$

$$\sigma_x = \frac{q_0}{b}\left\{ \begin{array}{l} -\frac{1}{10}\frac{z}{h}\frac{L^2}{h^2}\left[\begin{array}{l} \frac{120}{\pi^2}\left(1-2\frac{x}{L}-\cos\frac{\pi x}{L}\right) \\ -\frac{11520}{\pi^6}\frac{E}{G}\frac{h^2}{L^2}+\frac{120}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(1-\cos\frac{\pi x}{2L}\right) \end{array} \right] \\ +\frac{48}{\pi^4}\sin\frac{\pi z}{h}\frac{E}{G}\left(-\cos\frac{\pi x}{L}+\cosh\lambda x-\sinh\lambda x\right) \end{array} \right\} \tag{23}$$

$$\tau_{zx}^{CR} = \frac{48}{\pi^3}\frac{q_0}{b}\frac{L}{h}\cos\frac{\pi z}{h}\left(\frac{2}{\pi^2}-\frac{1}{\pi}\sin\frac{\pi x}{L}+\frac{\sinh\lambda x-\cosh\lambda x}{\lambda L}\right) \tag{24}$$

$$\tau_{zx}^{EE} = \frac{q_0 L}{80bh}\left(4\frac{z^2}{h^2}-1\right)\left[\frac{120}{\pi^2}\left(\pi\sin\frac{\pi x}{L}-2\right)+\frac{120}{\pi^2}\frac{E}{G}\frac{h^2}{L^2}\left(\pi\sin\frac{\pi x}{2L}\right)\right]$$
$$+\frac{48}{\pi^5}\cos\frac{\pi z}{h}\frac{E}{G}\frac{q_0}{b}\frac{h}{L}\left(\pi\sin\frac{\pi x}{L}+\lambda L\left(\sinh\lambda x-\cosh\lambda x\right)\right) \tag{25}$$

4. PERFORMANCE ANALYSIS

In this chapter, the results for inplane displacement, transverse displacement, inplane and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work.

For beam subjected to load, *q(x)*

$$q_0(x) = q_0\cos\frac{\pi x}{L}$$

$$\bar{u} = \frac{Ebu}{q_0 h}, \bar{\mathrm{w}} = \frac{10Ebh^3}{q_0 L^4}, \bar{\sigma}_x = \frac{b\sigma_x}{q_0}, \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q_0} \tag{26}$$

Table II: Non-Dimensional Axial Displacement $\bar{u}$ at (x = 0.25L, z), Transverse Deflection $(\bar{w})$ at (x = 0.25L, z) Axial Stress $(\bar{\sigma}_x)$ at (x = 0.25L, z), Maximum Transverse Shear Stresses $(\bar{\tau}_{zx}{}^{CR})$ and $(\bar{\tau}_{zx}{}^{EE})$ (x = 0, z = 0.0) of the Simply Supported Beam Subjected to cosine load for aspect ratio 4.

Model	$\bar{u}$	$(\bar{w})$	$(\bar{\sigma}_x)$	$(\bar{\tau}_{zx}^{CR})$	$(\bar{\tau}_{zx}^{EE})$
TSDT	-0.1121	1.3067	-2.2033	1.1570	-5.3908
FSDT	-0.0624	1.3067	-2.0145	0.5111	1.2159
ETB	-0.0624	0.7676	-2.0145	---	1.2159

Table II: Non-Dimensional Axial Displacement $\bar{u}$ at (x = 0.25L, z), Transverse Deflection $(\bar{w})$ at (x = 0.25L, z) Axial Stress $(\bar{\sigma}_x)$ at (x = 0.25L, z), Maximum Transverse Shear Stresses $(\bar{\tau}_{zx}^{CR})$ and $(\bar{\tau}_{zx}^{EE})$ (x = 0, z = 0.0) of the Simply Supported Beam Subjected to cosine load for aspect ratio 10.

Model	$\bar{u}$	$(\bar{w})$	$(\bar{\sigma}_x)$	$(\bar{\tau}_{zx}^{CR})$	$(\bar{\tau}_{zx}^{EE})$
TSDT	-1.0991	1.3067	-12.7794	3.0415	-3.5670
FSDT	-0.9747	1.3067	-12.5906	3.1945	3.0396
ETB	-0.9747	0.7676	-12.5906	---	3.0396

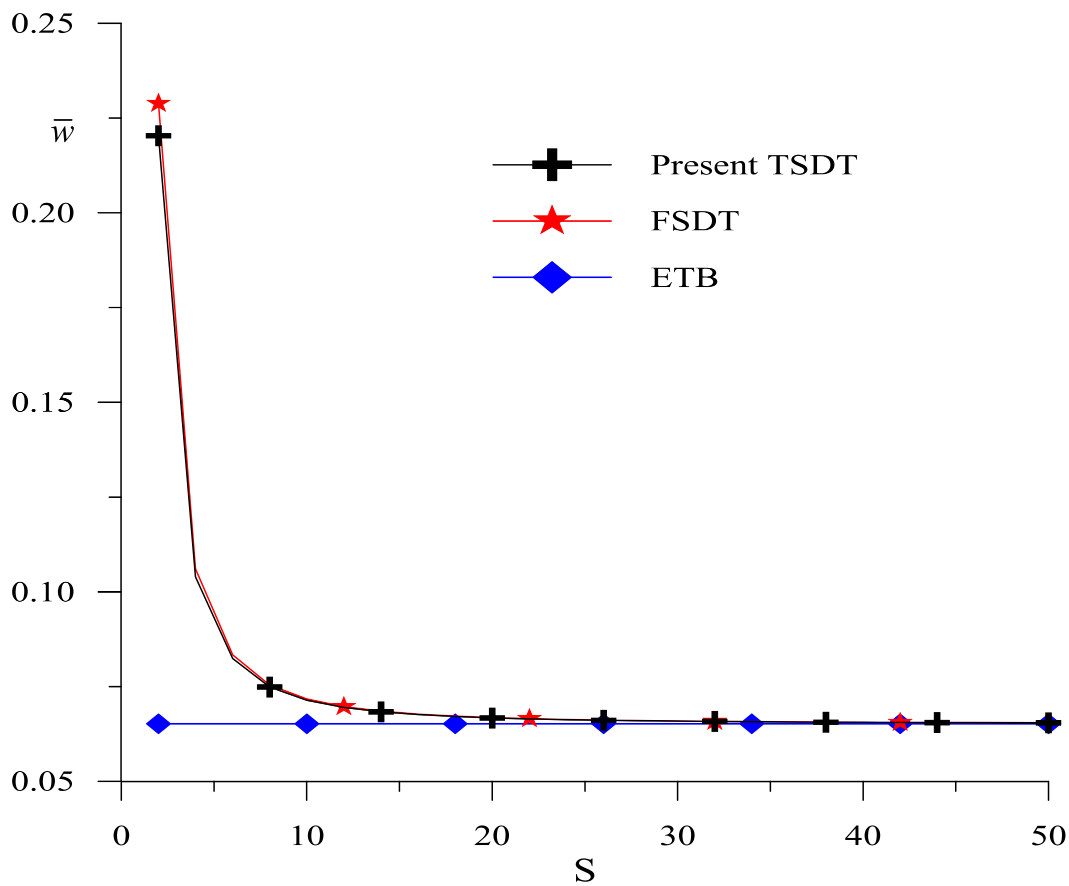

Fig. 3: Variation of maximum transverse displacement ($\bar{w}$) through the thickness of simply supported beam at (x = 0.25L, z) when subjected to cosine load for aspect ratio S.

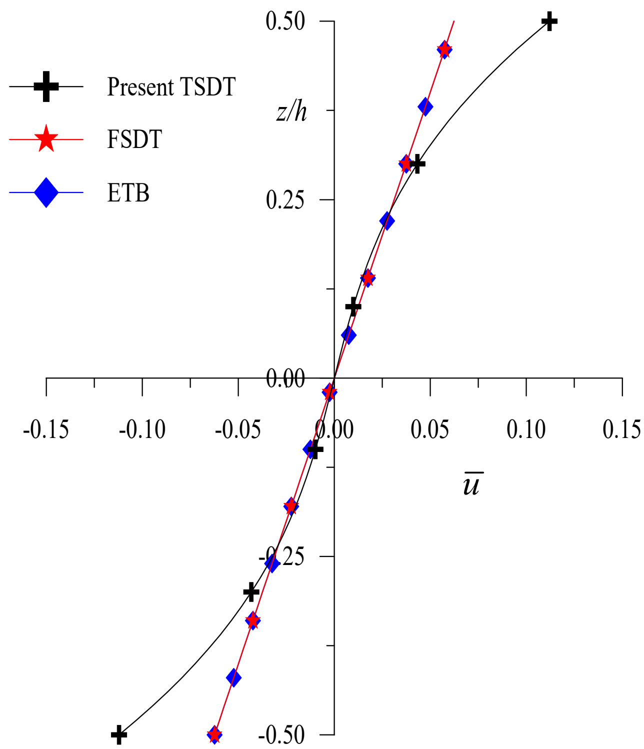

Fig. 4: Variation of axial displacement $\overline{u}$ through the thickness of simply supported beam at $(x = 0.25L, z)$ when subjected to cosine load for aspect ratio 4.

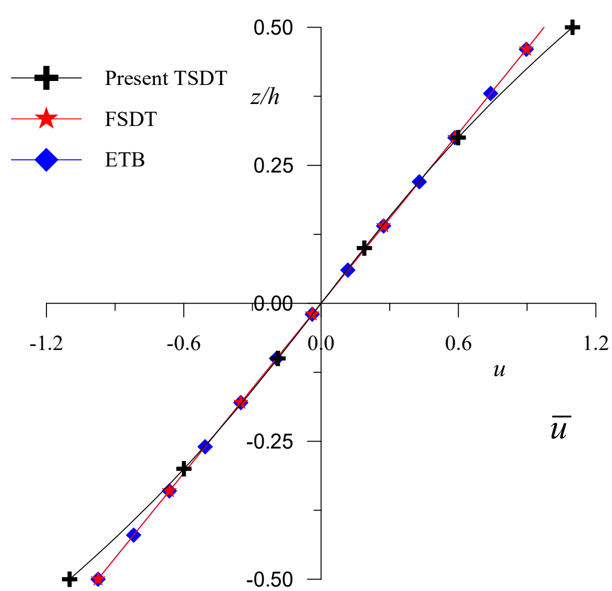

Fig. 5: Variation of axial displacement $\overline{u}$ through the thickness of simply supported beam at $(x = 0.25L, z)$ when subjected to cosine load for aspect ratio 10

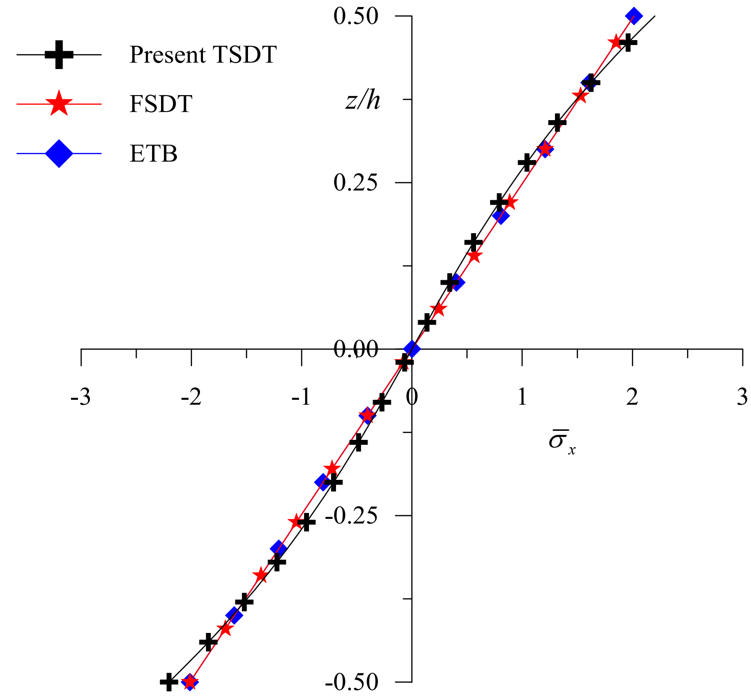

Fig. 6: Variation of axial stress $\left(\bar{\sigma}_x\right)$ through the thickness of simply supported beam at (x = 0.25L, z) when subjected to cosine load for aspect ratio 4.

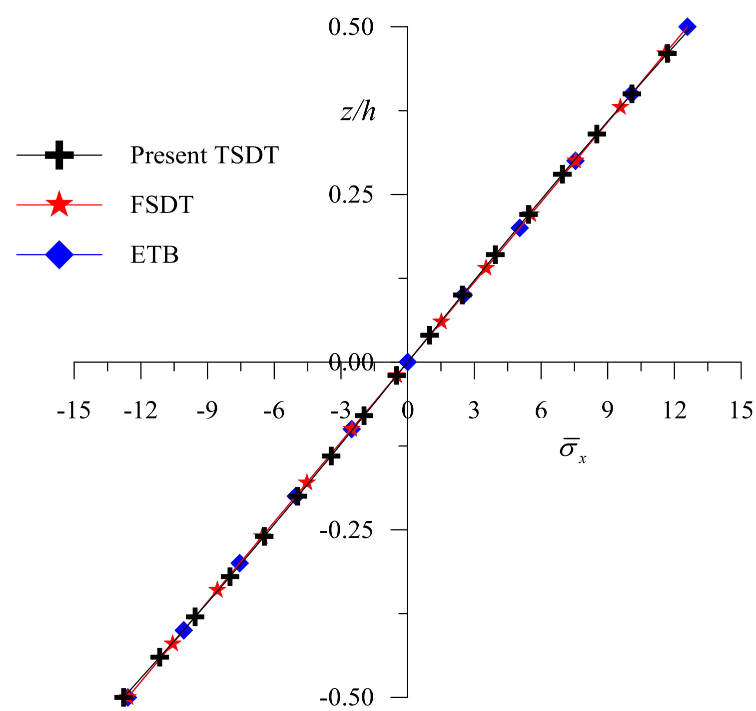

Fig. 7: Variation of axial stress $\left(\bar{\sigma}_x\right)$ through the thickness of simply supported beam at (x = 0.25L, z) when subjected to cosine load for aspect ratio 10.

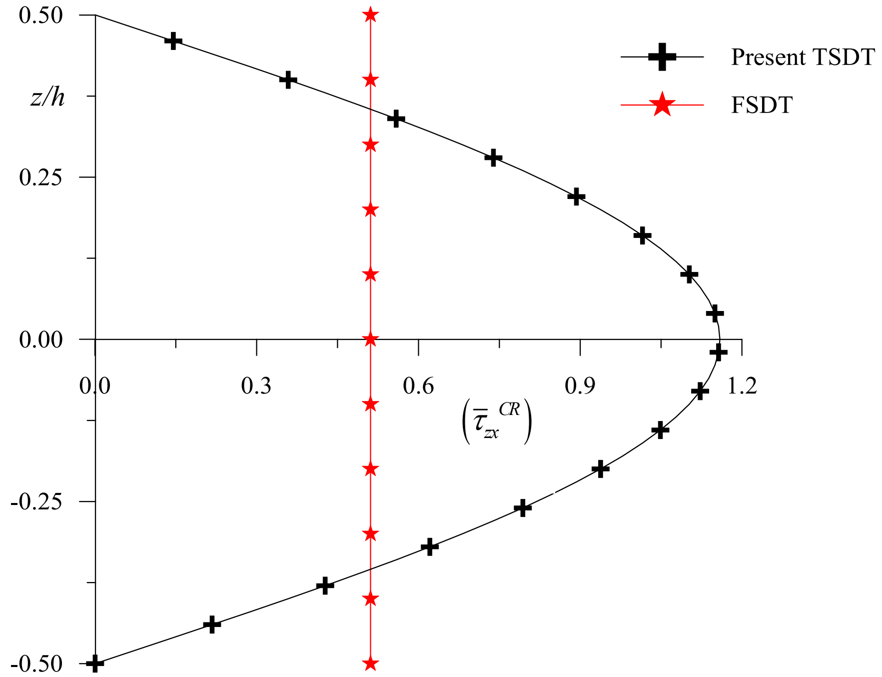

Fig. 8: Variation of transverse shear stress $\left(\bar{\tau}_{zx}^{CR}\right)$ through the thickness of simply supported beam at (x = 0, z) when subjected to cosine load and obtain using constitutive relation for aspect ratio 4.

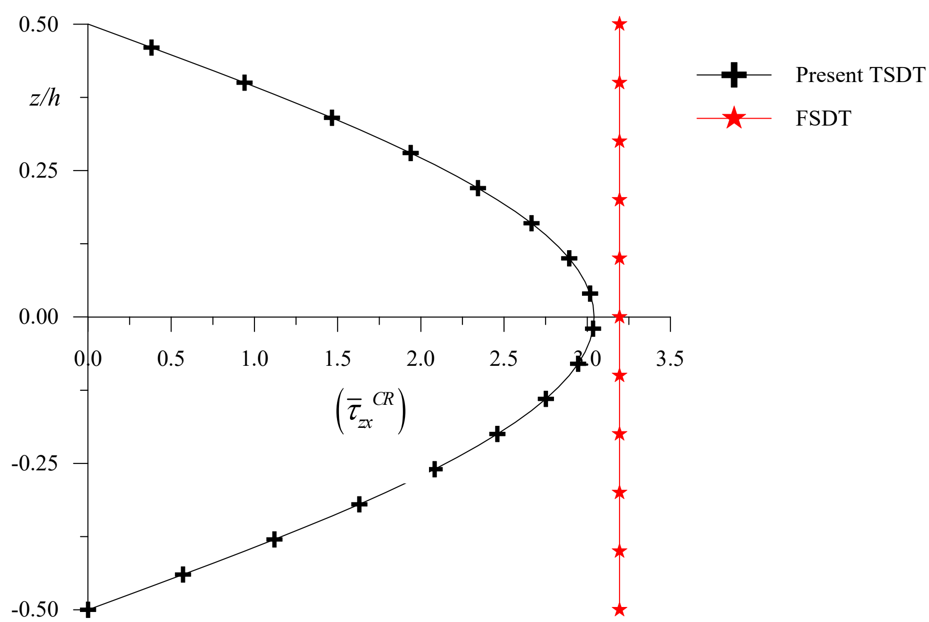

Fig. 9: Variation of transverse shear stress $\left(\bar{\tau}_{zx}^{CR}\right)$ through the thickness of simply supported beam at (x = 0, z) when subjected to cosine load and obtain using constitutive relation for aspect ratio 10.

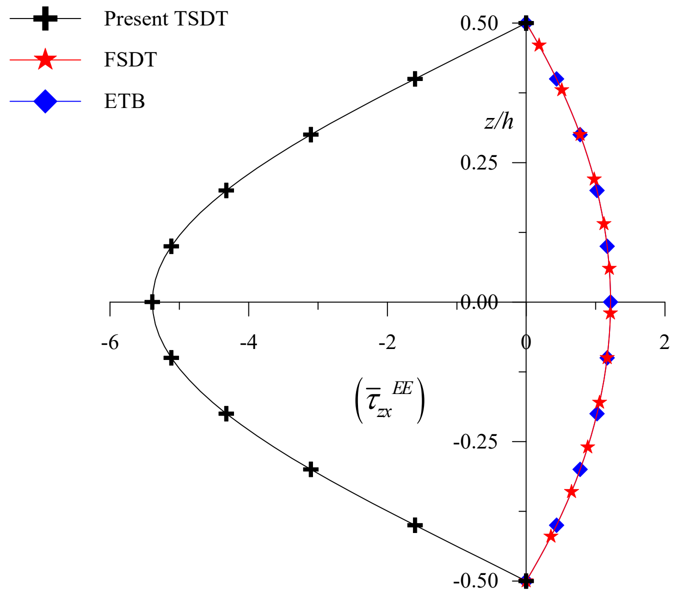

Fig. 10: Variation of transverse shear stress $\left(\overline{\tau}_{zx}^{EE}\right)$ through the thickness of simply supported beam at (x = 0, z) when subjected to cosine load and obtain using constitutive relation for aspect ratio 4.

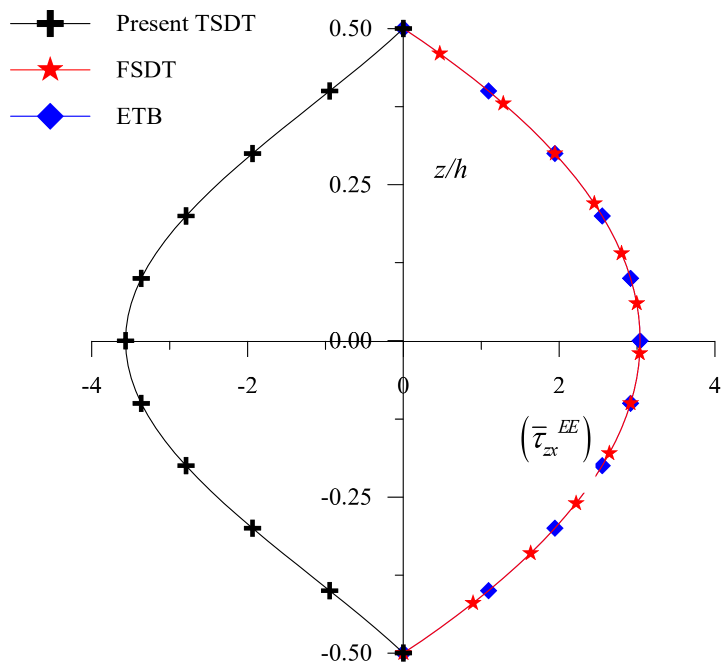

Fig. 11: Variation of transverse shear stress $\left(\overline{\tau}_{zx}^{EE}\right)$ through the thickness of simply supported beam at (x = 0, z) when subjected to cosine load and obtain using constitutive relation for aspect ratio 10.

5. DISCUSSION OF RESULTS

The comparison of results of maximum non-dimensional axial displacement $(\bar{u})$ for the aspect ratios of 4 and 10 is presented in Tables I and II for simply supported beams subjected to cosine load (see Figure 2). The values of axial displacement given by present theory are in close agreement with the values of other refined theories for aspect ratio 4 and 10. The through thickness distribution of this displacement obtained by present theory is in close agreement with other refined theories as shown in Figures 4 and 5 for aspect ratio 4 and 10.

The comparison of result of maximum non-dimensional transverse displacement $(\bar{w})$ is presented in Table I and II for simply supported beams subjected to cosine load. The value of present theory are in excellent agreement with the values of other refined theories except those of classical beam theory (ETB) and (FSDT) of Timoshenko. The variation of $(\bar{w})$ with aspect ratio (S) is shown in Figure 3. The refined theories converge to the value of classical beam theory for higher aspect ratios.

The results of axial stress $(\bar{\sigma}_x)$ are shown in Tables I and II for aspect ratios 4 and 10. The axial stresses given by present theory are compared with other higher order shear deformation theories. It is observed that the results by present theory are in excellent agreement with other refined theories as well as ETB and FSDT. The through the thickness variation of this stress given by all the theories is linear at $x = 0.25L$. The variations of this stress are shown in Figures 6 and 7. The comparison of maximum non-dimensional transverse shear stress for simple beams with cosine load obtained by the present theory and other refined theories is presented in Tables I and II for aspect ratio of 4 and 10 respectively. The maximum transverse shear stress obtained by present theory using constitutive relation is in good agreement with that of higher order theories for aspect ratio 4 and for aspect ratio 10. The through thickness variation of this stress obtained via constitutive relation are presented graphically in Figures 8 and 9 and those obtained via equilibrium equation are presented in Figures 10 and 11. The through thickness variation of this stress when obtained by various theories via equilibrium equation shows the excellent agreement with each other. The maximum value of this stress occurs at the neutral axis.

REFERENCES

[1] J. A. C. Bresse, ―Cours de Mechanique Applique‖, Mallet- Bachelier, Paris, 1859.

[2] J. W. S. Lord Rayleigh, ―The Theory of Sound‖, Macmillan Publishers, London, 1877.

[3] S. P. Timoshenko, J. N. Goodier, ―Theory of Elasticity‖, Third International Edition, McGraw-Hill, Singapore. 1970.

[4] G. R. Cowper, ―The shear coefficients in Timoshenko beam theory‖, ASME Journal of Applied Mechanic, vol. 33, no. 2, 1966, pp. 335-340.

[5] G. R. Cowper, ―On the accuracy of Timoshenko beam theory‖, ASCE J. of Engineering Mechanics Division. vol. 94, no. EM6, 1968, pp. 1447-1453,

[6] M. Levinson, ―A new rectangular beam theory‖, Journal of Sound and Vibration, Vol. 74, No.1, 1981, pp. 81-87.

[7] W. B. Bickford, ―A consistent higher order beam theory‖, International Proceeding of Development in Theoretical and Applied Mechanics (SECTAM), vol. 11, 1982, pp. 137-150,

[8] L. W. Rehfield, P. L. N. Murthy, ―Toward a new engineering theory of bending: fundamentals‖, AIAA Journal,

vol. 20, no. 5, 1982, pp. 693-699.

[9] A. V. Krishna Murty, ―Towards a consistent beam theory‖, AIAA Journal, vol. 22, no. 6, 1984, pp. 811-816.

[10] M. H. Baluch, A. K. Azad, and M. A. Khidir, ―Technical theory of beams with normal strain‖, ASCE Journal of Engineering Mechanics, vol. 110, no. 8, 1984, pp. 1233- 1237.

[11] A. Bhimaraddi, K. Chandrashekhara, ―Observations on higher order beam Theory‖, ASCE Journal of Aerospace Engineering, vol. 6, no. 4, 1993, pp. 408-413,

[12] H. Irretier, ―Refined effects in beam theories and their influence on natural frequencies of beam‖, International Proceeding of Euromech Colloquium, 219, on Refined Dynamical Theories of Beam, Plates and Shells and Their Applications, Edited by I. Elishak off and H. Irretier, Springer-Verlag, Berlin, 1986, pp. 163-179.

[13] T. Kant, A. Gupta, ―A finite element model for higher order shears deformable beam theory", Journal of Sound and Vibration, vol. 125, no. 2, 1988, pp. 193-202.

[14] P. R. Heyliger, J.N. Reddy, ―A higher order beam finite element for bending and vibration problems‖, Journal of Sound and Vibration, vol. 126, no. 2, 1988. pp. 309-326.

[15] R. C. Averill, J. N. Reddy, ―An assessment of four-noded plate finite elements based on a generalized third order theory‖, International Journal of Numerical Methods in Engineering, vol. 33, 1992, pp. 1553-1572.

[16] J. N. Reddy, ―An Introduction to Finite Element Method‖. 2nd Ed., McGraw-Hill, Inc., New York, 1993.

[17] V. Z. Vlasov, U. N. Leont'ev, ―Beams, Plates and Shells on Elastic Foundations‖ Moskva, Chapter 1, 1-8. Translated from the Russian by A. Barouch and T. Plez Israel Program for Scientific Translation Ltd., Jerusalem, 1966.

[18] M. Stein, ―Vibration of beams and plate strips with three dimensional flexibility‖, ASME J. of Applied Mechanics, vol. 56, no. 1, 1989, pp. 228-231.

[19] Y. M. Ghugal, R. P. Shmipi, ―A review of refined shear deformation theories for isotropic and anisotropic laminated beams‖, Journal of Reinforced Plastics and Composites, vol. 20, no. 3, , 2001, pp. 255-272.

[20] Y. M. Ghugal, R. Sharma, ―A hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams‖, International Journal of Computational Methods, vol. 6, no. 4, 2009, pp. 585-604.

[21] P. M. Pankade, D. H. Tupe and S. B. Salve, Static Flexural Analysis of Thick Isotropic Beam Using Hyperbolic Shear Deformation Theory, International Journal of Engineering Research, Vol.5 Issue: Special 3, 2016, 565-571.