Intuitionistic Fuzzy Relational Equations with Minimal Solution

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Abstract:

In this paper, we study the sufficient condition for the existence of a minimal solution for the Intuitionistic Fuzzy relational equation $xA = b$, and extend it to Intuitionistic Fuzzy Matrix equation $XA = Y$.

Key words and phrases: Intuitionistic Fuzzy Matrix (IFM), Intuitionistic Fuzzy Set (IFS) and Intuitionistic Fuzzy relation.

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1.Introduction

Atanassov [3] developed the concept of Intuitionistic Fuzzy Sets (IFSs) analogous to Fuzzy set. Im et., al [5] studied the determinant of square Intuitionistic Fuzzy Matrices (IFMs). Ketty Peeva and Yordan Kyosev [6] solved problems in Intuitionistic Fuzzy relational calculus by treating IFM as a Cartesian product of membership and non-membership functions. Meenakshi and Gandhimathi have studied the regularity, idempotency, invertibility and symmetry of IFMs in terms of those of its membership and non-membership matrices in [2] and discussed the consistency of Intuitionistic fuzzy relational equations in [1].

Sriram and Murugadas [7,8] studied Intuitionistic fuzzy vector space over Intuitionistic Fuzzy algebra and in [9] obtained maximal and minimal solution for Intuitionistic fuzzy relational equation. Guo et., al [4] studied the sufficient conditions for the existence of minimal solutions for the fuzzy relational equation $xA = b$. In this paper, we find the sufficient condition for the existence of a minimal solution of Intuitionistic Fuzzy relational equations. Intuitionistic Fuzzy relational equation $xA = b$ means x is an unknown intuitionistic fuzzy vector with known IFM A and known Intuitionistic fuzzy vector b. Also, we extend this concept of a sufficient condition for the existence of a minimal solution to $XA = Y$ where A and Y are known IFM with unknown X. Throughout this paper $I_n = \{1,2,\dots,n\}$, the index set.

2.Preliminaries

The set of all IFMs of order $m \times n$ is denoted by $\mathcal{F}_{mn}$.

Definition 2.1 1 [3]An Intuitionistic Fuzzy Set (IFS) A in E (universal set) is defined as an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E\}$, where the functions: $\mu_A(x): E \to [0,1]$ and $\nu_A(x): E \to [0,1]$ define the membership and non-membership functions of the element $x \in E$ respectively and for every $x \in E: 0 \leq \mu_{A(x)} + \nu_A(x) \leq 1$.

For simplicity, we consider the pair $\langle x, x'\rangle$ as membership and non-member functions of an IFS with $x + x' \leq 1$.

Definition 2.2 2 *[3] For $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, define*

$\langle x,x'\rangle+\langle y,y'\rangle=\langle max\{x,y\},min\{x',y'\}\rangle\langle x,x'\rangle\langle y,y'\rangle=\langle min\{x,y\},max\{x',y'\}\rangle$
$\langle x,x'\rangle^c=\langle x',x\rangle$.

Definition 2.3 3[7] Let $X=\{x_1,x_2,\dots x_m\}$ be the set of alternatives and $Y=\{y_1,y_2,\dots y_n\}$ be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by $A=(\langle(x_i,y_j),\mu_A(x_i,y_j),\nu_A(x_i,y_j)\rangle)$ for $i=1,2\dots m$ and $j=1,2,\dots n$, where $\mu_A:X\times Y\to[0,1]$ and $\nu_A:X\times Y\to[0,1]$ satisfy the condition $0\le\mu_A(x_i,y_j)+\nu_A(x_i,y_j)\le 1$. For simplicity, we denote an intuitionistic fuzzy matrix (IFM) as a matrix of pairs $A=(\langle a_{ij},a_{ij}'\rangle)$ of a nonnegative real numbers satisfying $a_{ij}+a_{ij}'\le 1$ for all i,j. We denote the set of all IFM of order $m\times n$ by $\mathcal{F}_{mn}$.

For any two elements $A=(\langle a_{ij},a_{ij}'\rangle),B=(\langle b_{ij},b_{ij}'\rangle)\in\mathcal{F}_{mn}$, define

1. $A\vee B=(\langle max\{a_{ij},b_{ij}\},min\{\ a_{ij}',b_{ij}'\}\rangle)=A\oplus B$,(component wise addition)

2. $A\wedge B=(\langle mn\{a_{ij},b_{ij}\},max\{a_{ij}',b_{ij}'\}\rangle)=AeB$,(component wise multiplication) for all
   $1\le i\le m$ and $1\le j\le n$.

3. $J=(\langle 1,0\rangle)$ the Universal matrix(matrix in which all entries are $\langle 1,0\rangle$)

4. $I=(\langle\delta_{ij},\delta_{ij}'\rangle)$(Identity Matrix) where $\langle\delta_{ij},\delta_{ij}'\rangle=\begin{cases}\langle 1,0\rangle & \text{if} i=j\\ \langle 0,1\rangle & \text{if} i\ne j\end{cases}$

5. $A\ge B$ if $a_{ij}\ge b_{ij}$ and $a_{ij}'\le b_{ij}'$ for all i,j.$A>B$ if $A\ge B,A\ne B$.(In this case A and B are comparable.)

6. $\overline{A}=(\langle a_{ij}',a_{ij}\rangle)$(complement of A).

Definition 2.44 [7] If $A=(\langle a_{ij},a_{ij}'\rangle)\in\mathcal{F}_{mn}$ and $B=(\langle b_{ij},b_{ij}'\rangle)\in\mathcal{F}_{np}$ then the product of A and B denoted as AB (max-min) is an IFM defined by

$AB=(\langle\max_{k=1}^{n}\{\min\{a_{ik},b_{kj}\}\},\min_{k=1}^{n}\{\max\{a_{ik}',b_{kj}'\}\}\rangle)$, where $1\le k\le n,1\le i\le m$ and $1\le j\le p$.

Definition 2.55 [7] Let $A=(\langle a_{ij},a_{ij}'\rangle)\in\mathcal{F}_{mn}$ and $c\in F=[1,0]$, then the intuitionistic fuzzy scalar multiplication is defined as $cA=(\langle min\{c,a_{ij}\},max\{1-c,a_{ij}'\}\rangle)\in\mathcal{F}_{mn}$.

Definition 2.66 [8] Let V_n denote the set of all n-tuples.

$(\langle v_1,v_{1'}\rangle,\dots\langle v_n,v_{n'}\rangle)$. The following operations are defined for

$v=(\langle v_1,v_{1'}\rangle,\dots\langle v_n,v_{n'}\rangle),s=(\langle s_1,s_{1'}\rangle,\dots\langle s_n,s_{n'}\rangle)$ in V_n and $r\in F=[0,1]$

$(\langle v_1,v_{1'}\rangle,\dots\langle v_n,v_{n'}\rangle)+(\langle s_1,s_{1'}\rangle,\dots\langle s_n,s_{n'}\rangle)=$

$(\langle v_1,v_{1'}\rangle+\langle s_1,s_{1'}\rangle,\dots\langle v_n,v_{n'}\rangle+\langle s_n,s_{n'}\rangle)$ and

$r(\langle v_1,v_{1'}\rangle,\dots\langle v_n,v_{n'}\rangle)=(r\langle v_1,v_{1'}\rangle,\dots r\langle v_n,v_{n'}\rangle)$.

The members of V_n have the properties

1. $v+w=w+v$

2. $v+(w+u)=(v+w)+u$

3. $(ab)v=a(bv)$

4. $(a+b)v=av+bv$

5. $a(v+w) = av + aw$

6. $1v = v$

7. $v + O = O + v = v$

8. $Ov = vO = O$,

where $u, v, w \in V_n$ and $a, b \in F, O = (\langle 0,1\rangle, \ldots \langle 0,1\rangle), 1 = \langle 1.0\rangle$.

If we write a member of V_n as $1 \times n$ matrix, it is called a row vector. The isometric set of $n \times 1$ matrices is called column vectors, and denoted by V^n. For any result about V_n there exists a corresponding result about V^n. The system V_n together with these operations is called intuitionistic fuzzy vector space over F.

$D(V_n)$ denotes the set of all finite subsets of V_n and $|A|$ the cardinality of an element $A \in D(V_n)$.

Definition 2.77 [8] A subspace W of V_n is a subset such that $O \in W$ and $a + b \in W$ for any $a, b \in W$. A linear combination of elements of a set $S = \{\langle s_1, s_{1'}\rangle \ldots . \langle s_k, s_{k'}\rangle\} \in D(V_n)$ is a finite sum $\sum_{i=1}^{k} \alpha_i \langle s_i, s_{i'}\rangle$ where $\alpha_i, \in [0,1]$. The set of all the linear combinations of elements of S is called the span of S, denoted by $\langle S\rangle$.

It follows immediately that $\langle S\rangle$ is contained in every subspace W such that $S \subseteq W$.

If we denote by $(e_i, e_{i'}), i \in I_n$, the element of V_n having $\langle 1,0\rangle - i^{th}$ coordinates and $\langle 0,1\rangle$ otherwise, it is evident that $V_n = \langle E\rangle$, where $E = \{\langle e_1, e_{1'}\rangle, \ldots \langle e_n, e_{n'}\rangle\} \in D(V_n)$.

Let $x = (\langle x_{11}, x_{11}'\rangle, \ldots, \langle x_{1m}, x_{1m}'\rangle), b = (\langle b_{11}, b_{11}'\rangle, \ldots, \langle b_{1n}, b_{1n}'\rangle)$. For $A \in \mathcal{F}_{mn}$,

$xA = b$ we mean $\max_j \min \langle x_{1j}, x_{1j}'\rangle \langle a_{jk}, a_{jk}'\rangle = \langle b_{1k}, b_{1k}'\rangle$ for $j \in I_m$ and $k \in I_n$.

Denote $\Omega(A, b) = \{x | xA = b\}$ (The solution set of $xA = b$)

Theorem 2.88 [1] Let $xA = b$. If $\langle \max_j a_{jk}, \min_j a_{jk}'\rangle < \langle b_{1k}, b_{1k}'\rangle$ for some $k \in I_n$,then $\Omega(A, b) = \emptyset$.

Definition 2.99 [1] For the intuitionistic fuzzy relational equation $xA = b$, the solution set $\Omega(A, b) = \{x / xA = b\} \neq \emptyset$ if and only if

$\hat{x} = [\langle \hat{x}_{1j}, \hat{x}_{1j}'\rangle | j \in I_m]$ defined as

$\langle \hat{x}_{1j}, \hat{x}_{1j}'\rangle = \langle \min \sigma(a_{jk}, b_{ik}), \max \sigma'(a_{jk}', b_{1k}')\rangle$,where

$\sigma(a_{jk}, b_{1k}) = \begin{cases} b_{1k} & \text{if } a_{jk} > b_{1k} \\ 1 & otherwise \end{cases}$ $\sigma'(a_{jk}', b_{1k}') = \begin{cases} b_{1k}' & \text{if } a_{jk}' < b_{1k}' \\ 0 & otherwise \end{cases}$ is the maximum solution of $xA = b$.

We can write the above definition equivalently as

$$\langle \hat{x}_{1j}, \hat{x}_{1j}'\rangle = \langle a_{jk}, a_{jk}'\rangle \to \langle b_{1k}, b_{1k}'\rangle = \begin{cases} \langle b_{1k}, b_{1k}'\rangle & \text{if} \langle a_{jk}, a_{jk}'\rangle > \langle b_{1k}, b_{1k}'\rangle \\ \langle 1,0\rangle & otherwise \end{cases}$$

which is the implication operator defined for IFS and IFM by Sriram and Murugadas [10] and in [9] Sriram and Murugadas have proved that the maximum solution $\langle \hat{x}, \hat{x'}\rangle$ for the intuitionistic fuzzy relation equation is unique and the minimal solution $\langle \breve{x}, \breve{x'}\rangle$ is not unique. The maximum and minimum solutions are related by $\langle \hat{x}, \hat{x'}\rangle \geq \langle \breve{x}, \breve{x'}\rangle$.

3. Some Results

Definition 3.110 Let $A, B \in D(V_n)$. B is called the spanning set of A if $A \subseteq \langle B \rangle$. Let $A \in D(V_n)$ and $\mathfrak{U} = \mathfrak{U}(A)$ the family of all the spanning sets of A. Note that $A \in \mathfrak{U}$ since $A \subseteq \langle A \rangle$. Hence $\mathfrak{U} \neq \emptyset$.

Definition 3.211 We say that an element $B \in \mathfrak{U}$ is a basis of A if $|B| = \omega(A)$, where $\omega(A) = min\{|C|: C \in \mathfrak{U}\}$.

Since $A \in \mathfrak{U}$ and clearly $E \in \mathfrak{U}$. Because $A \subseteq V_n = \langle E \rangle$, we have that $\omega(A) \leq min\{|A|, n\}$. Henceforth, we consider only subsets A of $D(V_n)$ such that $|A| \leq n$.

Now let $\langle y, y' \rangle \in V_n$ and $A \in D(V_n)$ be such that $\langle y, y' \rangle \in \langle A \rangle$. Let us consider the family $\mathfrak{B} = \mathfrak{B}(\langle y, y' \rangle, A)$ of all subsets $B \in D(V_n)$ contained in A and such that $\langle y, y' \rangle \in \langle B \rangle$. Note that $\mathfrak{B} \neq \emptyset$ since $A \in \mathfrak{B}$.

Definition 3.312 An element $B \in \mathfrak{B}$ is said to be a basis of $\langle y, y' \rangle$ in A if $|B| = \omega(\langle y, y' \rangle, A)$, where $\omega(\langle y, y' \rangle, A) = min\{|C|: C \in \mathfrak{B}\}$.

The basis B of $\langle y, y' \rangle$ in A is not unique, it is proved in the following example.

Example 3.4 Let $A = \{\langle a_1, a_{1'} \rangle, \langle a_2, a_{2'} \rangle, \langle a_3, a_{3'} \rangle, \langle a_4, a_{4'} \rangle\}$ where

$\langle a_1, a_{1'} \rangle = (\langle 0.1,0.9 \rangle, \langle 0.4,0.6 \rangle, \langle 0.5,0.5 \rangle, \langle 0.1,0.9 \rangle)$

$\langle a_2, a_{2'} \rangle = (\langle 0.9,0.1 \rangle), (\langle 0.7,0.3 \rangle, \langle 0.2,0.8 \rangle, \langle 0,1 \rangle)$

$\langle a_3, a_{3'} \rangle = (\langle 0.8,0.2 \rangle), (\langle 1,0 \rangle, \langle 0.5,0.5 \rangle, \langle 0,1 \rangle)$

$\langle a_4, a_{4'} \rangle = (\langle 0.1,0.9 \rangle), (\langle 0.3,0.7 \rangle, \langle 0.6,0.4 \rangle, \langle 0,1 \rangle)$ and

$\langle y, y' \rangle = (\langle 0.8,0.2 \rangle), (\langle 0.7,0.3 \rangle, \langle 0.5,0.5 \rangle, \langle 0,1 \rangle)$

It is easily seen that $\langle y, y' \rangle \in \langle A \rangle$ since

$\langle y, y' \rangle = 0\langle a_1, a_{1'} \rangle + 0.8\langle a_2, a_{2'} \rangle + 0.7\langle a_3, a_{3'} \rangle + 0.5\langle a_4, a_{4'} \rangle$ and

$B_1 = \{\langle a_2, a_{2'} \rangle, \langle a_3, a_{3'} \rangle\}, B_2 = \{\langle a_2, a_{2'} \rangle, \langle a_4, a_{4'} \rangle\}$ are such that

$\langle y, y' \rangle \in \langle B_i \rangle, i = 1,2.$

Since $\langle y, y' \rangle = 0.8\langle a_2, a_{2'} \rangle + 0.5\langle a_3, a_{3'} \rangle$ and $\langle y, y' \rangle = 0.8\langle a_2, a_{2'} \rangle + 0.5\langle a_4, a_{4'} \rangle$.

Also for any $\alpha \in [0,1], \langle y, y' \rangle \neq \alpha a_i$ for $i = 1,2,3,4$.

Hence $\omega(\langle y, y' \rangle, A) = 2$ and B_1, B_2 are distinct basis of $\langle y, y' \rangle$ in A.

Remark 3.5 If $\langle y, y' \rangle$ is not dependent on A, that is $\langle y, y' \rangle \notin \langle A \rangle$ we write $\omega(\langle y, y' \rangle, A) = 0$ and clearly $\omega(\langle y, y' \rangle, A) = 1$ if $\langle y, y' \rangle \in A$.

Theorem 3.613 If $B \in D(V_n)$ is a basis of $A \in D(V_n)$, then B is an independent set.

Proof: We have $|B| = \omega(A)$ and $A \subseteq \langle B \rangle$.

If B were dependent, then there would exist at least an intuitionistic fuzzy vector $\langle b, b' \rangle \in B$ such that $\langle b, b' \rangle \in \langle B \rangle$ where $B' = B - \{\langle b, b' \rangle\}$. Then $B \subseteq \langle B \rangle$ and this would imply $A \subseteq \langle B \rangle \subseteq \langle B' \rangle$, that is $B' \in A$. Thus $\omega(A) \leq |B'| = |B| - 1 = \omega(A) - 1$, a contradiction. Hence B is an independent set.

Similarly, we can prove that a basis$B$ of $y \in A$ is an independent set.

Theorem 3.714 Let V_n be the vector space on the intuitionistic fuzzy algebra $[0,1], \langle y,y'\rangle \in V_n$ and $A = \{\langle a_1, a_{1'}\rangle \dots, \langle a_q, a_{q'}\rangle\} \in D(V_n)$ such that $\langle y,y'\rangle \in \langle A\rangle$. If $\omega(A) < q \leq n$, then $\omega(\langle y,y'\rangle, A) < q \leq n$.

Proof: By putting $\omega(A) = k < q \leq n$,

Let $B = \{\langle b_1, b_{1'}\rangle, \dots \langle b_k, b_{k'}\rangle\}$ be a basis of A.

Then there exist $\lambda_{ij} \in [0,1], i \in I_n, j \in I_k$, such that

$\langle a_i, a_{i'}\rangle = \lambda_{i1}\langle b_1, b_{1'}\rangle + \lambda_{i2}\langle b_2, b_{2'}\rangle + \dots + \lambda_{ik}\langle b_k, b_{k'}\rangle$

$$\sum_{j=1}^{k} \lambda_{ij}\langle b_j, b_{j'}\rangle \tag{3.1}$$

for any $i \in I_n$.

Since $\langle y,y'\rangle \in \langle A\rangle$, then there exist $\alpha_i \in [0,1], i \in I_q$, such that

$\langle y,y'\rangle = \alpha_1\langle a_1, a_{1'}\rangle + \alpha_2\langle a_2, a_{2'}\rangle + \dots + \alpha_q\langle a_q, a_{q'}\rangle$

$= \alpha_1(\sum_{j=1}^{k} \lambda_{1j}\langle b_j, b_{j'}\rangle) + \dots + \alpha_q(\sum_{j=1}^{k} \lambda_{qj}\langle b_j, b_{j'}\rangle)$

$= (\sum_{i=1}^{q} \alpha_i\lambda_{i1})\langle b_1, b_{1'}\rangle + \dots + (\sum_{i=1}^{q} \alpha_i\lambda_{ik})\langle b_k, b_{k'}\rangle$

Since the sum of intuitionistic fuzzy algebra is the maximum, let $i(j)$ be an index such that $\sum_{i=1}^{q} \alpha_i\lambda_{ij} = \max_{1\leq i\leq q}\{\alpha_i\lambda_{ij}\} = \alpha_{i(j)}\lambda_{i(j)}, (3.2)$.

for any $j \in I_k$. Thus

$$\langle y,y'\rangle = (\alpha_{i(1)}.\lambda_{i(1)1})\langle b_1, b_{1'}\rangle + \dots + (\alpha_{i(k)}.\lambda_{i(k)k})\langle b_k, b_{k'}\rangle \tag{3.3}$$

We now prove the existence of a subset $B' \subseteq A$ such that $|B'| = k$ and $\langle y,y'\rangle \in \langle B'\rangle$.

Let $B' = \{\langle a_{i(1)}, a_{i(1)}{}'\rangle, \dots \langle a_{i(k)}, a_{i(k)}{}'\rangle\} \subseteq A$.

From (3.2) and (3.3)

$\alpha_{i(1)}\langle a_{i(1)}, a_{i(1)}{}'\rangle + \dots + \alpha_{i(k)}\langle a_{i(k)}, a_{i(k)}{}'\rangle$

$= \alpha_{i(1)}(\sum_{j=1}^{k} \lambda_{i(1)j}.\langle b_j, b_j{}'\rangle) + \dots + \alpha_{i(k)}(\sum_{j=1}^{k} \lambda_{i(k)j}.\langle b_j, b_j{}'\rangle)$

$= (\sum_{t=1}^{k} \alpha_{i(t)}\lambda_{i(t)1})\langle b_1, b_1{}'\rangle + \dots + (\sum_{t=1}^{k} \alpha_{i(t)}\lambda_{i(t)k})\langle b_k, b_k{}'\rangle)$

$= \alpha_{i(1)}\lambda_{i(1)1}\langle b_1, b_1{}'\rangle + \dots + \alpha_{i(k)}\lambda_{i(k)k}\langle b_k, b_k{}'\rangle$

$= \langle y,y'\rangle$

Then $\langle y,y'\rangle \in \langle B'\rangle$ and hence $B' \in \mathfrak{B}$. This implies $\omega(\langle y,y'\rangle, A) \leq k < q \leq n$ and hence the proof.

Let A be an IFM, $\langle a_i, a_{i'}\rangle = (\langle a_{i1}, a_{i1}{}'\rangle, \dots \langle a_{in}, a_{in}{}'\rangle)$ and

$\langle a^j, a^j{}'\rangle = (\langle a_{1j}, a_{1j}{}'\rangle, \dots \langle a_{nj}, a_{nj}{}'\rangle)$ be the i^{th} - row and the j^{th} -column, respectively of A.

Let $A_r = \{\langle a_1, a_{1'}\rangle, \dots \langle a_n, a_{n'}\rangle\}$

and $A^c = \{\langle a^1, a^{1'}\rangle, \dots, \langle a^n, a^{n'}\rangle\}$ be the sets of the row vectors and column vectors of A, respectively. Then $A_r, A^c \in D(V_n)$ and the following result holds.

Theorem 3.815 $\omega(A_r) = \omega(A^c)$.

Proof: Let $\omega(A_r) = k \leq n$ and $B = \{\langle b_1, b_{1'}\rangle, \ldots. \langle b_k, b_{k'}\rangle\}$ be a basis of A_r, where $\langle b_s, b_{s'}\rangle = (\langle b_{s1}, b_{s1}{}'\rangle, \ldots \langle b_{sn}, b_{sn}{}'\rangle) \in V_n$ for any $s \in I_k$. Then there exist $\lambda_{ij} \in [0,1], i \in I_n, j \in I_k$ such that (3.1) holds for any $i \in I_n$. Thus each element $\langle a_{ij}, a_{ij}{}'\rangle$ in $\langle a_i, a_i{}'\rangle$ can be written as

$$\langle a_{ij}, a_{ij}{}'\rangle = \sum_{s=1}^{k} \lambda_{is}\langle b_{sj}, b_{sj}{}'\rangle \text{ for any } i, j \in I_n \quad (3.4)$$

If we put $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ where $\lambda_s = \{\lambda_{1s}, \ldots, \lambda_{ns}\}$ for any $s \in I_k$ we deduce from (3.4) that

$\langle a^j, a^{j\prime}\rangle = \sum_{s=1}^{k} \langle b_{sj}, b_{sj'}\rangle\lambda_s$ for any $j \in I_n$. This means that $A^c \subseteq <\Lambda>$ and hence $\Lambda \in \mathfrak{U}(A^c)$.

Thus $\omega(A^c) \leq |\Lambda| = k$ and assume that $\omega(A^c) = p < k$. Thus there exist p vectors of V_n whose span contain A^c. Similarly we can prove any vector of A_r is a linear combination of p vectors of V_n, a contradiction to the assumption that $\omega(A_r) = k > p$.

4.Application of intuitionistic fuzzy relation equations:

Let $\Omega = \Omega(A, y) = \{$the set of all solutions of $/xA = y\}$ where $x = \langle x, x'\rangle = (\langle x_1, x_{1'}\rangle, \ldots, \langle x_n, x_{n'}\rangle) \in V_n, y = \langle y, y'\rangle = (\langle y_1, y_{1'}\rangle, \ldots., \langle y_n, y_{n'}\rangle) \in V_n$. Clearly $\Omega = \emptyset$ if and only if $\omega(y, A) = 0$ and $A \cup \{y\}$ is a dependent set of V_n if $\Omega \neq \emptyset$.

Theorem 4.116 Let $\Omega \neq \emptyset$. if $\omega(\langle y, y'\rangle, A) = n$, then for any $\langle x, x'\rangle \in \Omega$ we have $\langle x_i, x_{i'}\rangle > \langle 0,1\rangle$ for any $i \in I_n$.

Proof: Let $\langle x, x'\rangle \in \Omega$ be such that $\langle x_h, x_{h'}\rangle = \langle 0,1\rangle$ for some $h \in I_n$. Then

$$\langle y_j, y_{j'}\rangle = \langle x_1, x_{1'}\rangle\langle a_{1j}, a_{1j}{}'\rangle + ,\ldots., + \langle x_{h-1}, x_{h-1}{}'\rangle\langle a_{(h-1)j}, a_{(h-1)j}{}'\rangle + \langle a_{h+1}, a_{h+1}{}'\rangle\langle a_{(h+1)j}, a_{(h+1)j}{}'\rangle + ,\ldots., + \langle x_n, x_n{}'\rangle\langle a_{nj}, a_{nj}{}'\rangle \quad (4.1)$$

for any $j \in I_n$ and this means that $\langle y, y'\rangle$ is a linear combination of at most $n-1$ row vectors of A, that is $n = \omega(\langle y, y'\rangle, A) \leq n-1$, a contradiction.

Remark 4.217 Let $f = (f(1), f(2), \ldots, f(n))$ be a permutation of I_n. Let us consider the intuitionistic fuzzy vector $\langle y, y'\rangle = (\langle y_1, y_{1'}\rangle, \ldots.., \langle y_n, y_{n'}\rangle)$ such that $\langle y, y'\rangle = \langle y_{f(i)}, y_{f(i)}{}'\rangle$ for any $i \in I_n$ and the IFM $A' = \langle a_{ij}, a_{ij}{}'\rangle, i, j \in I_n$ such that $\langle a_{ij}, a_{ij}{}'\rangle = \langle a_{if(i)}, a_{if(i)}{}'\rangle$ for any $j \in I_n$. Now if $\langle x, x'\rangle$ is a solution of $\langle x, x'\rangle A = \langle y, y'\rangle$, (4.2)

we have $\langle y_j, y_{j'}\rangle' = \langle y_{f(i)}, y_{f(i)}{}'\rangle = \sum_{i=1}^{n} \langle x_i, x_{i'}\rangle\langle a_{if(i)}, a_{if(i)}\rangle$ that is $\langle x, x'\rangle$ is also a solution of the IF equation $\langle x, x'\rangle A' = \langle y, y'\rangle$. ....(4.3).

Similarly, let $\langle x, x'\rangle$ be a solution of (4.2), we have for any $j \in I_n$,

$$\langle y_j, y_{j'}\rangle = \langle y_{f^{-1}(i)}, y_{f^{-1}(j)}\rangle$$

$$= \sum_{i=1}^{n} \langle x_i, x_{i'}\rangle\langle a_{if^{-1}(j)}, a_{if^{-1}(j)}{}'\rangle'$$

$$= \sum_{i=1}^{n} \langle x_i, x_{i'}\rangle\langle a_{ij}, a_{ij}{}'\rangle,$$

which implies that $\langle x, x'\rangle$ is also solution of the IF equation $\langle x, x'\rangle A = \langle y, y'\rangle$. Then the equations (4.2) and (4.3) have the same solution and clearly every Theorem or Property concerning (4.2), concerns similarly (4.3) and vice versa.

Clearly, we have $\omega(\langle y, y'\rangle, A) = \omega(\langle y, y'\rangle, A')$.

Remark 4.318 It is evident that for $\langle x, x'\rangle \in \Omega$, there exist at least an index $i \in I_n$ such that $\langle x_i, x_{i'}\rangle\langle a_{ij}, a_{ij}'\rangle = \langle y_j, y_j'\rangle$ for any $j \in I_n$.

Definition 4.4 19 For each $\langle x, x'\rangle \in \Omega$, Let

$I_j(\langle x, x'\rangle) = \{i \in I_n : \langle x_i, x_{i'}\rangle\langle a_{ij}, a_{ij}'\rangle = \langle y_j, y_{j'}\rangle\}$ for any $j \in I_n$.

Theorem 4.520 Let $\Omega \neq \emptyset$. Then $\omega(\langle y, y'\rangle, A) = n$ if and only if for any $\langle x, x'\rangle \in \Omega$ we have $I_j(\langle x, x'\rangle) \cap I_{j'}(\langle x, x'\rangle) = \emptyset$ for any $j, j' \in I_n$.

Proof: Let $\omega(\langle y, y'\rangle, A) = n$ and without loss of generality let

$I_1(\langle x, x'\rangle) \cap I_2(\langle x, x'\rangle) = \{h\}$ for some $\langle x, x'\rangle \in \Omega$

Then we have

$\langle y_1, y_{1'}\rangle = \langle x_h, x_{h'}\rangle\langle a_{h1}, a_{h1}'\rangle$,

$\langle y_2, y_{2'}\rangle = \langle x_h, x_{h'}\rangle\langle a_{h2}, a_{h2}'\rangle$,

$\langle y_j, y_{j'}\rangle = \langle x_t, x_{t'}\rangle\langle a_{t1}, a_{t1}'\rangle$,

for any $j \in I_n - \{1,2\}$, where $t \in I_j(\langle x, x'\rangle)$.

It is easily seen that these inequalities imply that $\langle y, y'\rangle$ is a linear combination of at most $n-1$ row vectors of A, that is $n = \omega(\langle y, y'\rangle, A) \leq n-1$, a contradiction.

Assume $I_j(\langle x, x'\rangle \cap I_{j'}\langle x, x'\rangle) = \emptyset$ for any $j, j' \in I_n$ and $\langle x, x'\rangle \in \Omega$ implies

$|I_j(\langle x, x'\rangle)| = 1$ for any $j \in I_n$ and $\langle x, x'\rangle \in \Omega$.

Suppose $\Omega(\langle y, y'\rangle, A) = k < n$ and let

$B = \{\langle a_{i(1)}, a_{i(1)}'\rangle, \ldots \langle a_{i(k)}, a_{i(k)}'\rangle\}$ be a basis of $\langle y, y'\rangle$ in A. Since $\langle y, y'\rangle$ is dependent on B, there exist k elements $\lambda_{i(t)} \in [0,1], t \in I_k$, such that

$\langle y, y'\rangle = \langle \lambda_{i(1)}, \lambda_{i(1)}'\rangle\langle a_{i(1)}, a_{i(1)}'\rangle + \ldots + \langle \lambda_{i(k)}, \lambda_{i(k)}'\rangle\langle a_{i(k)}, a_{i(k)}'\rangle$, that is

$$\langle y_j, y_{j'}\rangle = \langle \lambda_{i(1)}, \lambda_{i(1)}'\rangle\langle a_{i(1)j}, a_{i(1)j}'\rangle + \ldots + \langle \lambda_{i(k)}, \lambda_{i(k)}'\rangle\langle a_{i(k)j}, a_{i(k)j}'\rangle \qquad (4.4)$$

for any $j \in I_n$.

Defining the fuzzy vector $\langle x, x'\rangle$ having the $i(t)^{th}$ co-ordinate, $t \in I_k$, equal to $\langle \lambda_{i(t)}, \lambda_{i(t)}'\rangle$ and $\langle 0,1\rangle$ otherwise it follows from (4.4) that $\langle x, x'\rangle \in \Omega$.

Since $|I_j(\langle x, x'\rangle)| = 1$ for any $j \in I_n$, equation (4.4) implies that for any $j \in J = \{i(t) : t \in I_k\}$,there exist a unique index $t \in I_k$ such that $\langle y_j, y_{j'}\rangle = \langle x_{i(t)}, x_{i(t)}'\rangle'\langle a_{i(t)j}, a_{i(t)j}'\rangle$

Let $j' \in I_n - J$. We have that the singleton set $I_{j'}(\langle x, x'\rangle)$ must necessarily be an index $i(t) \in J, t \in I_k$; otherwise if $I_{j'}(\langle x, x'\rangle) = \{h\}$ with $h \in I_n - J$,

we would obtain

$\langle 0,1\rangle < \langle y_{j'}, y'_{j'}\rangle = \langle x_h, x_{h'}\rangle\langle a_{hj}, x_{hj}{}'\rangle = \langle 0,1\rangle$ a contradiction.

On the other hand $\{i(t)\} = I_j(\langle x, x'\rangle)$ for some $j \in J$ and hence

$I_j(\langle x, x'\rangle) \cap I_{j'}(\langle x, x'\rangle) = \{i(t)\}$, a contradiction to the hypothesis.

By Theorem 4.5, $I_j(\langle x, x'\rangle)$ is a singleton set for any $j \in I_n$ and for any $\langle x, x'\rangle \in \Omega$ if $\omega(\langle y, y'\rangle, A) = n$. However, we explicitly point out that for any $h \in I_n$, there exists a unique index $j \in I_n$ such that $I_j(\langle x, x'\rangle) = \{h\}$ for any $\langle x, x'\rangle \in \Omega$.

Theorem 4.621 Let $\Omega \neq \emptyset$. If $\omega(\langle y, y'\rangle, A) = n$ then for any $j \in I_n$ we have $I_j(\langle x, x'\rangle) = I_j(\langle x, x'\rangle)$ for any $\langle x, x'\rangle, \langle x, x'\rangle \in \Omega$.

Proof: Let $\langle x, x'\rangle \in \Omega$ and for any $j \in I_n$ denote by $i(j)$ the unique element of I_n

such that $\langle y, y_{j'}\rangle = \langle x_{i(j)}, x_{i(j)}{}'\rangle\langle a_{i(j)j}, a_{i(j)j}{}'\rangle$. (4.5)

Thus $I_j(\langle x, x'\rangle) = \{i(j)\}$ and $\langle x, x'\rangle$ is another element of Ω, we must prove that

$I_j(\langle x, x'\rangle) = \{i(j)\}$ for any $j \in I_n$.

Assume that $\langle y_j, y_{j'}\rangle > \langle x_{i(j)}, x_{i(j)}{}'\rangle'\langle a_{i(j)j}, a_{i(j)j}{}'\rangle$ (4.6)

and let $k \in I_n$ such that

$\langle y_k, y_{k'}\rangle = \langle x_{i(j)}, x_{i(j)}{}'\rangle'\langle a_{i(j)k}, a_{i(j)k}{}'\rangle$ (4.7)

clearly $j \neq k$ and $i(j) \neq i(k)$ by theorem (4.2)

Then we have $\langle y_k, y_{k'}\rangle > \langle x_{i(j)}, x_{i(j)}{}'\rangle\langle a_{i(j)k}, a_{i(j)k}{}'\rangle$ (4.8)

From (4.5) $\langle a_{i(j)j}, a_{i(j)j}{}'\rangle \geq \langle y_j, y_{j'}\rangle$ (4.9)

We claim $\langle y_j, y_j{}'\rangle > \langle x_{i(j)}, x_{i(j)}{}'\rangle'$. (4.10)

Suppose if, $\langle y_i, y_{i'}\rangle \leq \langle x_{i(j)}, x_{i(j)}{}'\rangle'$ and (4.9) if follows that

$\langle y_j, y_{j'}\rangle \leq \langle x_{i(j)}, x_{i(j)}{}'\rangle'\langle a_{i(j)j}, a_{i(j)j}{}'\rangle$ a contradiction to (4.6).

By (4.7), $\langle x_{i(j)}, x_{i(j)}\rangle \geq \langle y_k, y_{k'}\rangle$

Thus from (4.10) $\langle y_j, y_{j'}\rangle \geq \langle y_k, y_{k'}\rangle$ (4.11)

On the other hand , from (4.5) $\langle x_{i(j)}, x_{i(j)}\rangle \geq \langle y_j, y_{j'}\rangle$ and by (4.7)

$\langle a_{i(j)k}, a_{i(j)k}{}'\rangle \geq \langle y_k, y_{k'}\rangle$

Thus we obtain from (4.8) and (4.11)

$\langle y_k, y_{k'}\rangle > \langle x_{i(j)}, x_{i(j)}{}'\rangle\langle a_{i(j)k}, a_{i(j)k}{}'\rangle \geq \langle y_j, y_j{}'\rangle\langle y_k, y_k{}'\rangle = \langle y_k, y_k{}'\rangle$

a contradiction, this means $j = k$. Hence the Theorem.

Remark 4.7 22 Theorem (4.6) guarantees that

$\langle x_i, x_i{}'\rangle'\langle a_{ij}, a_{ij}{}'\rangle = \langle y_j, y_j{}'\rangle$ (4.12)

holds for any $\langle x, x'\rangle' \in \Omega - \{\langle x, x'\rangle\}$. In account of this, it is evident that if $\langle a_{ij}, a_{ij}{}'\rangle = \langle y_j, y_i{}'\rangle$ the greatest value $\langle \hat{x}_i, \hat{x}_i{}'\rangle$ to put in $\langle x_i, x_i'\rangle$ in order to satisfy (4.12) is

$\langle 1,0 \rangle$, while the smallest value $\langle \breve{x}_i, \breve{x}_i' \rangle$ is equal to $\langle y_i, y_i' \rangle$. If $\langle a_{ij}, a_{ij}' \rangle > \langle y_j, y_j' \rangle$, then the unique value to put in $\langle x_i, x_i' \rangle$ in order to satisfy the equality $\langle x_i, x_i' \rangle \langle a_{ij}, a_{ij}' \rangle = \langle y_j, y_j' \rangle$ is equal to $\langle y_j, y_j' \rangle$. Thus the following result holds.

Theorem 4.723 Let $\Omega \neq 0$. If $\omega(\langle y, y' \rangle, A) = n$, then Ω has a minimum element $\langle \breve{x}, \breve{x}' \rangle$. We illustrate this with the following example.

Example 4.824 Let $n = 2, \langle y, y' \rangle = (\langle 0.5,0.3 \rangle, \langle 0.4,0.5 \rangle)$ and A by $A = \begin{bmatrix} \langle 0.5,0.3 \rangle & \langle 0.3,0.5 \rangle \\ \langle 0.8,0.1 \rangle & \langle 0.5,0.1 \rangle \end{bmatrix}$, we have $\langle x, x' \rangle = (\langle 1,0 \rangle \langle 0.4,0.5 \rangle)$ is a solution of $\langle x, x' \rangle A = \langle y, y' \rangle$. Therefore $\Omega \neq \emptyset$. The smallest element is $\langle \breve{x}, \bar{\breve{x}}' \rangle = (\langle 0.5,0.3 \rangle, \langle 0.4,0.5 \rangle)$. Further $I_1(\langle \breve{x}, \bar{\breve{x}}' \rangle) = \{1\}$ and $I_2(\langle \breve{x}, \bar{\breve{x}}' \rangle) = \{2\}$. ByTheorem (4.5) $\omega(\langle y, y' \rangle, n) = 2$.

Remark 254.9 The condition is not necessary, it is illustrated through the following Example.

Example 4.10 Let $n = 2, \langle y, y' \rangle = (\langle 0.8,0.2 \rangle, \langle 0.5,0.3 \rangle)$ and $A = \begin{bmatrix} \langle 0.8,0.1 \rangle & \langle 0.5,0.4 \rangle \\ \langle 0.6,0.4 \rangle & \langle 0.5,0.4 \rangle \end{bmatrix}$. Since $(\langle 0.8,0.2 \rangle, \langle 1,0 \rangle) A = \langle y, y' \rangle, \Omega \neq \emptyset$. Further, $(\langle 0.8,0.2 \rangle, \langle 0,1 \rangle)$ is the minimum element of Ω and since the second co-ordinate equal to $\langle 0,1 \rangle$, we have $\Omega(\langle y, y' \rangle, A) < 2$ by Theorem 4.1.

Again, $0.6(\langle 0.8,0.1 \rangle, \langle 0.5,0.4 \rangle) = (\langle 0.6,0.4 \rangle, \langle 0.5,0.4 \rangle) = (\langle 0.6,0.4 \rangle, \langle 0.5,0.4 \rangle)$

Therefore condition given in Theorem 4.7 is not necessary.

The following example shows that the results can be extended to Intuitionistic Fuzzy matrix equation $XA = Y$, where A and Y are known IFMs with unknown IFM X.

Example 4.11 Let $XA = Y$ with $A = \begin{bmatrix} \langle 0.5,0.3 \rangle & \langle 0.3,0.5 \rangle \\ \langle 0.8,0.1 \rangle & \langle 0.5,0.1 \rangle \end{bmatrix}, Y = \begin{bmatrix} \langle 0.5,0.3 \rangle & \langle 0.4,0.5 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.5,0.2 \rangle \end{bmatrix}$ using Definition 2.9, the maximal solution is $\hat{X} = \begin{bmatrix} \langle 1,0 \rangle & \langle 0.4,0.5 \rangle \\ \langle 1,0 \rangle & \langle 0.5,0.2 \rangle \end{bmatrix}$. Let $\hat{X}_i, Y_i$be the i^{th} rows of $\hat{X}, Y$. Consider $\hat{X}_1 A = Y_1, \hat{X}_2 A = Y_2$, as like in Example 4.10 we can find the smallest elements$\breve{X}_1 = (\langle 0.5,0.3 \rangle, \langle 0.4,0.5 \rangle)$ and $\breve{X}_2 = (\langle 0.5,0.2 \rangle, \langle 0.5,0.2 \rangle)$ and hence the smallest element is $\breve{X} = \begin{bmatrix} \langle 0.5,0.3 \rangle & \langle 0.4,0.5 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.5,0.2 \rangle \end{bmatrix}$.

Conclusion

In this work only sufficient condition for the existence of minimal solution is presented, work for necessary and sufficient condition for the existence of minimal solution is in progress.

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