Approximating The Fixed Points of Multivalued ρ-Quasi-nonexpansive Mapping for Picard-Mann Hybrid Iterative Scheme

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ABSTRACT: *The aim of this research paper is to introduce Picard-Mann hybrid iterative scheme to approximate the fixed points of multivalued ρ - quasi-nonexpansive mappings in modular function spaces.*

KEYWORDS: Fixed point, multivalued ρ -quasi- nonexpansive, Picard-Mann hybrid iterative scheme.

INTRODUCTION

The concept of order spaces was generalized by Nakano [13] to the modular spaces in 1950. Then it was further generalized and redefined by Musielak and Orlicz [12]. As the modular function spaces are the generalization of some class of Banach spaces, so many analysts show their interest in working in this field in modular function spaces. Khamsi, Kozlowski and Reich [7] were the first who initiated the study of fixed point theory in these spaces in 1990. On the basis of their results, many work has been done in these spaces.

In 2014, Khan and Abbas [8] were the first who gave the approximation theorem for fixed points of a multivalued ρ -nonexpansive mapping in these spaces by using Mann iteration scheme. In this paper, we make an attempt to approximate the fixed points of ρ -quasi-nonexpansive multivalued mappings in these spaces for Picard-Mann hybrid iteration scheme.

BASIC DEFINITIONS

Let Ω be a nonempty set and Σ be a nontrivial σ - algebra of subsets of Ω. Let Ƥ be a nontrivial δ-ring of subsets of Ω which means that Ƥ is closed under countable intersection, and finite union and differences. Suppose that E∩A ∈ Ƥ for any E ∈ Ƥ and A ∈ Σ. Let us assume that there exists an increasing sequence of sets K_n ∈ Ƥ such that Ω = ∪K_n. By Ɛ we denote the linear space all simple functions with support from Ƥ. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions f: Ω → [-∞, ∞] such that there exists a sequence

$$\{g_n\}\subset \mathcal{E},\ |g_n| \le |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathcal{M} = \{ f \in \mathcal{M}_\infty : |f(w)| < \infty\ \rho - a.e. \}$$

DEFINITION 2.1[9] Let X be a vector space (R or C). A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary f and g, elements of X, there hold the following:

i) $\rho(f)=0 \Leftrightarrow f=0$

ii) $\rho(\alpha f)=\rho(f)$ whenever $|\alpha|=1$

iii) $\rho(\alpha f+\beta g)\leq \rho(f)+\rho(g)$ whenever $\alpha,\ \beta\geq 0,\ \alpha+\beta=1$

If we replace (iii) by

iv) $\rho(\alpha f+\beta g)\leq \alpha\,\rho(f)+\beta\rho(g)$ whenever $\alpha,\beta\geq 0,\ \alpha+\beta=1$

Then modular ρ is called convex.

DEFINITION 2.2[9] If ρ is convex modular in X, then the set defined by

$$L_\rho=\{f\in \mathrm{M} : lim_{\lambda\to 0}\rho(\lambda f)=0\}$$

is called modular function space.

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F- norm defined by

$$\| f \|_\rho = inf\{a>0:\rho\left(\frac{f}{a}\right)\leq a\}.$$

In the case ρ is convex modular,

$$\| f \|_\rho = inf\{a>0:\rho\left(\frac{f}{a}\right)\leq 1\}$$

defines a norm on modular space L_ρ and it is called Luxemburge norm.

DEFINITION 2.3[9] Let $\rho:\mathrm{M}_\infty\to[0,\ \infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudomodular if

(1) $\rho(0)=0;$

(2) ρ is monotone, i.e., $|f(w)|\leq|g(w)|$ for any $w ò \Omega$ implies $\rho(f)\leq\rho(g)$, where $f,g\in \mathrm{M}_\infty$;

(3) ρ is orthogonally subadditive, i.e., $\rho(f\chi_{A\cup B})\leq\rho(f\chi_A)+\rho(f\chi_B)$ for any A, B ∈ ∑ such that $A\cap B\neq\phi,\ f\in \mathrm{M}_\infty$;

(4) ρ has Fatou property, i.e., $|f_n(w)|\uparrow|f(w)|$ for $w ò \Omega$ implies $\rho(f_n)\uparrow\rho(f)$, where $f\in \mathrm{M}_\infty$;

(5) ρ is order continuous in Ɛ, i.e., g_n ∈ Ɛ, and $|g_n(w)|\downarrow 0$ implies $\rho(g_n)\downarrow 0.$

A set A ∈ ∑ is said to be ρ -null if $\rho(g\chi_A)=0$ for every g ∈ Ɛ. A property $p(w)$ is said to hold ρ - almost everywhere $(\rho-a.e.)$ if the set $\{w\in\Omega: p(w)\,does\ not\ hold\}$ is ρ -null.

DEFINITION 2.4[9] A regular function pseudomodular ρ is a regular convex function modular if $\rho(f)=0$ implies $f=0$ $a.e.$ The class of all nonzero regular convex function modular defined on Ω will be denoted by $\Re$.

DEFINITION 2.5 [9] Let $\rho \in \Re$.

(1) A sequence $\{f_n\}$ is $\rho-$convergent to f, that is, $f_n \to f$ if $\rho(f_n - f) \to 0$ as $k \to \infty$.

(2) A sequence $\{f_n\}$ in is called $\rho-$Cauchy sequence if $\rho(f_n - f_m) \to 0$ as n, m → ∞.

(3) A set $B \subset L_\rho$ is called $\rho-$closed if for any sequence $\{f_n\} \subset B$, the convergence $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(4) A set $B \subset L_\rho$ is called $\rho-$bounded if its $\rho-$diameter is finite; the $\rho-$diameter of B is defined as $\delta_\rho(B) = \sup\{\rho(f-g) : f, g \in B\}$.

(5) A set $B \subset L_\rho$ is called $\rho-$compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.

(6) A set $B \subset L_\rho$ is called $\rho-a.e.$ closed if for any sequence $\{f_n\} \subset B$ which $\rho-a.e.$ converges $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(7) A set $B \subset L_\rho$ is called $\rho-a.e.$ compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0 a.e.$ as $k \to \infty$.

(8) Let $f \in L\rho$ and $B \subset L_\rho$. The distance between f and B is defined as

$$d_\rho(f, B) = inf\{\rho(f-g) : g \in B\}.$$

PROPOSITION 2.6[9] Let $\rho \in \Re$.

(i) L_ρ is $\rho-$complete.

(ii) $\rho-$balls $B_\rho(f,r) = \{g \in L_\rho : \rho(f-g) \le r\}$ are $\rho-$closed.

(iii) If $\rho(\alpha f_n) \to 0$ for $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ $\rho-a.e.$ as $n \to \infty$.

(iv) $\rho(f) \le lim\ inf_{n\to\infty} \rho(f_n)$ whenever $f_n \to f$ $\rho-a.e.$ as $n \to \infty$. (Note: this property is equivalent to the Fatou property.)

(v) Consider the set $L_\rho^0 = \{f \in L_\rho : \rho(f, .)\ is\ order\ continuous\}$ and,

$E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0\ for\ \text{any}\ \lambda > 0\}$. Then we have $E_\rho \subset L_\rho^0 \subset L_\rho$.

DEFINITION 2.7[9] Let $\rho \in \Re$. Then ρ satisfies Δ_2 - property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to \infty$.

PROPOSITION 2.8[9] The following statements are equivalent:

(i) ρ satisfies Δ_2 – condition.

(ii) $\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for every $\lambda > 0$ if and only if $\| f_n - f \|_\rho \to 0$ as $n \to 0$.

DEFINITION 2.9[8] A set $C \subset L_\rho$ is called $\rho-$proximinal if for each $f \in L_\rho$, there exists an element g ∈ C such that

$$\rho(f - g) = d_\rho(f, C) = inf(\rho(f - h) : h \in C).$$

$P_\rho(C)$ denotes the family of nonempty ρ-bounded ρ-proximininal subset of C and $C_\rho(C)$ denotes the family of $\rho-$bounded $\rho-$closed subsets of C. Let $H_\rho(.,.)$ be $\rho-$Hausdorff distance on $C_\rho(L_\rho)$, that is,

$$H_\rho(A, B) = max\{sup_{f \in A}\, dist\rho(f, B),\ sup_{g \in B}\, dist_\rho(g, A)\},\ A,\ B \in C_\rho(L_\rho).$$

DEFINITION 2.10[8] A multivalued mapping $T : C \to C_\rho(L_\rho)$ is said to be $\rho-$quasi-nonexpansive

$$H_\rho(T(f), p) \le \rho(f - p) \text{ for all } f, g \in C$$

LEMMA 2.11[2] Let ρ ò $\Re$ and satisfy (UUC1). Let $\{t_n\} \subset (0,1)$ be bounded away from both 0 and 1. If there exists R > 0 such that

$$lim_{n\to\infty} sup\rho(f_n) \le R, lim_{n\to\infty} sup\rho(g_n) \le R \ and$$
$$lim_{n\to\infty}(t_n f_n + (1 - t_n)\ g_n) = R,\ then$$
$$lim_{n\to\infty}\rho(f_n - g_n) = 0$$

The sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from 0 if there exists a > 0 such that $t_n \ge a$ for all n ∈ N. Similarly the sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from 1 if there exists b < 1 such that $t_n \le b$ for all n ∈ N.

LEMMA 2.12[8] Let T: C→ $P_\rho(C)$ be a multivalued mapping and

$$P^T{}_\rho(f) = \{g \in T : \rho(f - g) = d_\rho(f,\ Tf)\}$$

Then the following are equivalent:

i) $f \in F_\rho(T)$, that is $f \in T(f)$;

ii) $P^T_\rho(f) = \{f\}, i.e., f = g$ for each $g \in P^T_\rho(f)$;

iii) $f \in F_\rho(P_\rho^T(f))$ that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T)$ where $F(P_\rho^T)$ denotes the set of fixed points of P_ρ^T.

LEMMA 2.13: Let ρ ò $\Re$ and satisfy Δ_2 –condition $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then

$$lim_{n\to\infty}\rho(g_n) = 0 \ implies \ lim_{n\to\infty} sup\, \rho(f_n + g_n) = lim_{n\to\infty} sup\, \rho(f_n)$$

and

$$lim_{n\to\infty}\rho(g_n) = 0 \ implies \ lim_{n\to\infty} inf\, \rho(f_n + g_n) = lim_{n\to\infty} inf\, \rho(f_n).$$

LEMMA 2.14: Let ρ ò $\Re$ and A, B ∈ $P_\rho(L_\rho)$. For every f ∈ A, there exists g ∈ B such that

$$\rho(f - g) \le H_\rho(A, B)$$

MAIN RESULTS

We prove some approximation theorems for Picard-Mann hybrid iterative scheme which is more general than that of the Mann iterative scheme used by Safeer Husain Khan and Mujahid Abbas [6]. This iterative scheme is as follows:

Let C $\subset L_\rho$ be a non empty $\rho-$bounded, closed and convex set and $T : C \to P_\rho(C)$ be a multivalued mappings. Let $f_1 \in C$ and { f_n }⊂ C be defined by

$$f_{n+1} \in P_\rho^T(g_n)$$
$$g_n = \alpha_n u_n + (1-\alpha_n) f_n \qquad n = 1, 2, \ldots \qquad (3.1)$$

where $u_n \in P_\rho^T(f_n)$ and $\{\alpha_n\}$ is sequence in (0,1) which are bounded away from both 0 and 1.

Before proving our main results, we firstly prove the following Lemma:

Lemma 3.1: Let ρ ò $\Re$ and C ⊂ L_ρ be nonempty ρ-bounded and convex set. Suppose T : C → P_ρ(C) be a multivalued mappings such that P_ρ^T is ρ -quasi-nonexpansive mapping with F (T) ≠ ϕ. Then the $lim_{n\to\infty}\rho(f_n - p)$ exists for all p ∈ F and $lim_{n\to\infty}\rho(f_n - u_n) = 0.$

Proof Let p ∈ F(T) be arbitrary. Then by Lemma 2.12, we have

$$P_\rho^T(p) = \{p\}$$

Now from eq. (3.1), we have

$$\rho\left(f_{n+1} - p \right) \le H_\rho\left(P_\rho^T(g_n), p \right) \le \rho(g_n - p) \quad (3.2)$$

$$\rho(g_n - p) = \rho(\alpha_n u_n + (1-\alpha_n) f_n - p)$$
$$\le \alpha_n \rho\left(u_n - p \right) + (1-\alpha_n)\, \rho\left(f_n - p \right)$$
$$\le \alpha_n H_\rho(P_\rho^T\left(f_n \right), p) + (1-\alpha_n)\, \rho\left(f_n - p \right)$$
$$\le \alpha_n \rho\left(f_n - p \right) + (1-\alpha_n)\, \rho\left(f_n - p \right)$$
$$= \rho\left(f_n - p \right) \quad (3.3)$$

Then from eq, (3.2) and (3.3), we obtained that

$$\rho\left(f_{n+1} - p \right) \le \rho\left(f_n - p \right)$$

Therefore, the sequence { $\rho(f_n - p)$ } is decreasing. Hence

$lim_{n\to\infty} \rho\left(f_n - p \right)$ exists for all p ∈ $F_\rho(T)$.

Let

$$\lim_{n\to\infty} \rho(f_n - p) = R$$

(3.4)

From eq. (3.3) and eq. (3.4), we get

$$\lim_{n\to\infty} \sup \rho(g_n - p) \le R$$

(3.5)

This implies

$$\rho\left(u_n - p \right) \le \rho(P_\rho^T\left(f_n \right) - p)$$
$$\le \rho\left(f_n - p \right)$$

which implies that

$$lim_{n\to\infty} \sup \rho\left(u_n - p \right) \le lim_{n\to\infty} \sup \rho\left(f_n - p \right)$$

or $$lim_{n\to\infty} \sup \rho\left(u_n - p \right) \le R \quad (3.6)$$

Since the sequence $\{\alpha_n\} \subset (0, 1)$ is bounded away from 0 and 1, there exists $\alpha \in (0, 1)$ such that

$$lim_{n\to\infty} \alpha_n = \alpha$$

Now,

$$\rho\left(f_{n+1} - p \right) = \rho(\alpha_n u_n + (1-\alpha_n) f_n - p)$$
$$= \rho(\alpha_n \left(u_n - p \right) + (1-\alpha_n)\left(f_n - p \right))$$

$$\leq \alpha_n \rho(u_n - p) + (1-\alpha_n)\, \rho(f_n - p)$$

$$lim_{n\to\infty} inf\ \rho(f_{n+1} - p) \leq lim_{n\to\infty} inf\ (\alpha_n \rho(u_n - p) + (1-\alpha_n)\, \rho(f_n - p))$$

$$\leq\ lim_{n\to\infty} inf\ (\alpha_n \rho(u_n - p)) + lim_{n\to\infty} inf\ (1-\alpha_n)\, \rho(f_n - p)$$

or
$$R \leq \alpha lim_{n\to\infty} inf\ \rho(u_n - p) + (1-\alpha)\, R$$

which implies that

$$R \leq lim_{n\to\infty} inf\ \rho(u_n - p)$$

(3.7)

From eq. (3.6) and eq. (3.7), we get

$$lim_{n\to\infty} \rho(u_n - p) = R \qquad (3.8)$$

Since $\lim_{n\to\infty} \rho(f_{n+1} - p) = R$

This implies

$$lim_{n\to\infty} \rho(\alpha_n u_n + (1-\alpha_n)\, f_n - p) = R$$

or
$$lim_{n\to\infty} \rho(\beta_n (u_n - p) + (1-\beta_n)(f_n - p)) = R \qquad (3.9)$$

Then from eq. (3.4), eq. (3.7), eq. (3.9) and Lemma (2.12), we obtain that

$$lim_{n\to\infty} \rho(f_n - u_n) = 0 \qquad (3.10)$$

or
$$lim_{n\to\infty} dis_\rho \rho(f_n, P_\rho^T(f_n)) = 0$$

Theorem 3.2: Let $\rho \in \Re$ satisfy (UUC1) and C ⊂ L_ρ be nonempty $\rho-$bounded and convex set. Let T : C → P_ρ(C) be a multivalued mapping such that P_ρ^T is ρ-quasi-nonexpansive mapping with $F_\rho(T)$ ≠ ϕ. Let f_1 ∈ C and { f_n } be given by (3.1). Then the sequence { f_n } converges to a fixed point of T.

Proof: Using the compactness of C, there must be a subsequence { f_{n_k} } of { f_n } and f ∈ C such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$. We will show that f is a fixed point of T, i.e., f ∈ $F_\rho(T)$. Let $g \in P_\rho^T(f)$ be arbitrary. Then by Lemma 2.16, $g_k \in P_\rho^T(f_{n_k})$ such that

$$\rho(g_k - g) \leq H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f))$$

We have

$$\rho\left(\frac{f-g}{3}\right)=\rho\left(\frac{f-f_{n_k}}{3}+\frac{f_{n_k}-g_k}{3}+\frac{g_k-g}{3}\right)$$
$$\leq\frac{1}{3}\rho(f-f_{n_k})+\frac{1}{3}\rho(f_{n_k}-g_k)+\frac{1}{3}\rho(g_k-g)$$
$$\leq\rho(f-f_{n_k})+dis_{\rho}(f_{n_k}-P_{\rho}^{T}(f_{n_k}))+\rho(g_k-g)$$
$$\leq\rho(f-f_{n_k})+dis_{\rho}(f_{n_k}-P_{\rho}^{T}(f_{n_k}))+H_{\rho}(P_{\rho}^{T}(f_{n_k}),P_{\rho}^{T}(f))$$
$$\leq\rho(f-f_{n_k})+d_{\rho}(f_{n_k}-P_{\rho}^{T_1}(f_{n_k}))+\rho(f-f_{n_k})\to 0$$
as k→∞.

Hence $f=g$ a.e. Since $g\in P_{\rho}^{T}(f)$ was arbitrary, we have $P_{\rho}^{T}(f)$ = { f }. Thus by using Lemma 2.12, $f\in T(f)$) and thus $f\in F_{\rho}(T)$. This completes the proof.

REFERENCES

[1] Dehaish, B.A.B. and Kozlowski, W.M. 2012. Fixed point iteration processes for asymptotically pointwise nonexpansive mapping in modular function spaces. Fixed Point Theory Appl., 2012, Article ID 118.

[2] Dhomopongsa, S., Benavides, T.D., Kaewcharoen A. and Panyanak B. 2006. Fixed point theorem for multivalued mappings in modular function spaces. Sci. Math. Jpn., 2006, 139-147.

[3] Ishikawa S. 1974. Fixed points by a new iteration method. Proc. Amer. Math. Soc., 44, 147-150.

[4] Khamsi, M.A. and Kozlowski W.M. 2011. On asymptotic pointwise nonexpansive mappings in modular function spaces. J. Math. Anal. Appl., 2011, 697-708.

[5] Khamsi, M.A., Kozlowski, W.M. and Reich, S. 1990. Fixed point theory in modular function spaces. Nolinear Anal., 14, 935-953.

[6] Khan, S.H. and Abbas, M. 2014. Approximating the fixed points of multivalued ρ-nonexpansive mappings in modular function spaces. Fixed Point Theory Appl., 2014, 1-9.

[7] Kozlowski, W.M. 1988. Modular function spaces. Monograph and Textbook in Pure and Applied Mathematics, 122, Marcel Dakker, New York, USA, 1988.

[8] Kutib, M.A. and Latif A. 2009. Fixed points of multivalued mappings in modular function spaces. Fixed Point Theory Appl., 2009, Artcle 786357.

[9] Mann W.R. 1953. Mean value methods in iteration. Proc. Amer. Math. Soc., 44, 506-510.

[10] Musielak, J. and Orlicz, W. 1959. On modular spaces, Stud. Math., 18, 591-597.

[11] Nakano, H. 1950. Modular semi- ordered spaces, Maruzen, Tokyo, 1950.