Approximation of Fixed Points of Multivalued ρ-Nonexpansive Mappings for Picard-Ishikawa Hybrid Iterative Scheme

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ABSTRACT: *In this paper, we approximate the fixed point of multivalued ρ -nonexpansive mappings in modular function spaces by using Picard-Ishikawa hybrid iterative scheme.*

KEYWORDS: Fixed point, multivalued ρ - nonexpansive, Picard-Ishikawa hybrid iteration scheme, modular function spaces.

INTRODUCTION

In 1950, Nakano [13] generalized the order spaces in terms of the modular spaces. After that Musielak and Orlicz [12] redefined these spaces. Actually modular function spaces are the generalization of some class of Banach spaces, so many analysts show their interest in working in this field. Khamsi, Kozlowski and Reich [7] were the first who initiated the study of fixed point theory in these spaces in 1990. By using their results, many researchers worked in these spaces.

In 2014, Khan and Abbas [8] were the first who gave the approximation theorem for fixed points of a multivalued ρ -nonexpansive mapping in these spaces by using Mann iteration scheme.Recently, Okeke G.A. et al. [14] gave some convergence results to approximate the fixed point of multivalued ρ -quasi-nonexpansive mappings using Picard Krasnoselski hybrid iterative scheme in modular function spaces. Motivated by this, we will prove some convergence results for ρ -nonexpansive mappings using Picard-Ishikwa hybrid iterative scheme in these spaces. Our results will extend, generalize and improve many results, including the recent results of Khan et al. [11] and Okeke G.A. [14].

BASIC DEFINITIONS

Let Ω be a nonempty set and Σ be a nontrivial σ - algebra of subsets of Ω. Let Ƥ be a nontrivial δ-ring of subsets of Ω which means that Ƥ is closed under countable intersection, and finite union and differences. Suppose that $E \cap A \in \mathrm{P}$ for any $E \in \mathrm{P}$ and A∈ Σ. Let us assume that there exists an increasing sequence of sets $K_n \in$ Ƥ such that $\Omega = \cup K_n$. By ξ we denote the linear space all simple functions with support from Ƥ. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions f: $\Omega \to [-\infty,\infty]$ such that there exists a sequence

$$\{g_n\} \subset \xi,\ |g_n| \le |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathrm{M} = \{ f \in \mathrm{M}_\infty : |f(w)| < \infty \rho - a.e.\}$$

DEFINITION 2.1[9]: Let X be a vector space (R or C). A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary f and g, elements of X, there hold the following:

i) $\rho(f) = 0 \Leftrightarrow f = 0$

ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$

iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ whenever $\alpha,\ \beta \ge 0,\ \alpha + \beta = 1$

If we replace (iii) by

iv) $\rho(\alpha f + \beta g) \le \alpha\, \rho(f) + \beta \rho(g)$ whenever $\alpha,\ \beta \ge 0,\ \alpha + \beta = 1$

Then modular ρ is called convex.

DEFINITION 2.2[9]: If ρ is convex modular in X, then the set defined by

$$L_\rho = \{ f \in \mathrm{M} : lim_{\lambda \to 0} \rho(\lambda f) = 0\}$$

is called modular function space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\| \mathrm{f} \|_\rho = \inf \left\{ a > 0, \rho\left(\frac{f}{a} \right) \le a \right\}$$

In the case ρ is convex modular,

$$\| \mathrm{f} \|_\rho = \inf \left\{ a > 0, \rho\left(\frac{f}{a} \right) \le 1 \right\}$$

defines a norm on modular space L_ρ and it is called Luxemburge norm.

DEFINITION 2.3: Let ρ be a nonzero regular convex function modular defined on Ω.

$$D_1(r,\ \varepsilon) = \left\{ (f,\ g) : f, g \in L_\rho,\ \rho(f) \le r,\ \rho(g) \le r, \rho(f - g) \ge \varepsilon r \right\}$$

Let

$$\delta_1(r, \varepsilon) = inf \left\{ 1 - \frac{1}{r} \rho\left(\frac{f + g}{2} \right) : (f,\ g) \in D_1(r,\ \varepsilon) \right\}$$

$$if\ D_1(r,\ \varepsilon)\ \ne\ \phi, and\, \delta_1(r, \varepsilon) = 1.$$

DEFINITION 2.4: A non-zero regular convex function modular ρ is said to satisfy (UC1) if every $r > 0,\ \varepsilon > 0,\ \delta_1(r,\ \varepsilon) > 0.$ Note that for every $r > 0,\ D_1(r, \varepsilon) \ne \phi$ for $\varepsilon > 0$ small enough.

DEFINITION 2.5: A non-zero regular convex function modular ρ is said to satisfy (UUC1) if for every $s \geq 0$, $\varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending only upon s and ε such that $\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0$ for any $r > s$.

DEFINITION 2.6[9]: Let $\rho \in \Re$.

(1) A sequence $\{f_n\}$ is ρ -convergent to f that is, $f_n \to f$ if $\rho(f_n - f) \to 0$ as $k \to \infty$.

(2) A sequence $\{f_n\}$ in is called ρ - Cauchy sequence if $\rho(f_n - f_m) \to 0$ as n, m → ∞.

(3) A set $B \subset L_\rho$ is called ρ -closed if for any sequence $\{f_n\} \subset B$, the convergence $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(4) A set $B \subset L_\rho$ is called ρ -bounded if its $\rho-$ diameter is finite; the ρ -diameter of B is defined as $\delta_\rho(B) = \sup\{\rho(f - g) : f, g \in B\}$.

(5) A set $B \subset L_\rho$ is called $\rho-$ compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.

(6) A set $B \subset L_\rho$ is called $\rho - a.e.$ closed if for any sequence $\{f_n\} \subset B$ which $\rho - a.e.$ converges $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(7) A set $B \subset L_\rho$ is called $\rho - a.e.$ compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0 a.e.$ as $k \to \infty$.

(8) Let $f \in L_\rho$ and $B \subset L_\rho$. The distance between f and B is defined as

$$d_\rho(f, B) = inf\{\rho(f - g) : g \in B\}.$$

PROPOSITION 2.7[9]: Let $\rho \in \Re$.

(i) L_ρ is $\rho-$ complete.

(ii) $\rho-$ ball $B_\rho(f,r) = \{g \in L_\rho : \rho(f - g) \leq r\}$ are $\rho-$ closed.

(iii) If $\rho(\alpha f_n) \to 0$ for $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$g_n \to 0$ $\rho - a.e.$ as $n \to \infty$.

(iv) $\rho(f) \leq lim\ inf_{n\to\infty} \rho(f_n)$ whenever $f_n \to f$ $\rho - a.e.$ as $n \to \infty$. (Note: this property is equivalent to the Fatou property.)

(v) Consider the set

$$L_\rho^0 = \{f \in L_\rho : \rho(f, .) \text{ is order continuous}\} and E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0 \text{ for any } \lambda > 0\}.$$

Then we have, $E_\rho \subset L_\rho^0 \subset L_\rho$.

DEFINITION 2.8[9]: Let $\rho \in \Re$. Then ρ satisfies Δ_2- property if $\rho(2f_n)\to 0$ whenever $\rho(f_n)\to 0$ as $n\to\infty$.

PROPOSITION 2.9[9]: The following statements are equivalent:

(i) ρ satisfies Δ_2- condition.

(ii) $\rho(f_n - f)\to 0$ if and only if $\rho(\lambda(f_n - f))\to 0$, for every $\lambda > 0$ if and only if $\| f_n - f \|_\rho \to 0$ as $n \to 0$.

DEFINITION 2.10[8]: A set $C\subset L_\rho$ is called $\rho-$ proximinal if for each $f\in L_\rho$, there exists an element g ∈ C such that

$$\rho(f-g)=d_\rho(f,C)=inf(\rho(f-h):h\in C).$$

$P_\rho(C)$ denotes the family of nonempty ρ-bounded ρ-proximininal subset of Cand $C_\rho(C)$denotes the family of $\rho-$bounded $\rho-$closed subsets of C. Let $H_\rho(.,\ .)$ be $\rho-$ Hausdorff distance on $C_\rho(L_\rho)$, that is,

$$H_\rho(A,B)=max\{sup_{f\in A}\, dist\rho(f,B),\ sup_{g\in B}\, dist_\rho(g,A)\},\ A,\ B\in C_\rho(L_\rho).$$

DEFINITION 2.11[8]: A multivalued mapping $T:C\to C_\rho(L_\rho)$ is said to be $\rho-$ Lipschitizian if there exists a number k ≥ 0 such that

$$H_\rho(T(f),T(g))\le k\rho(f-g) \text{ for all } f,g\in C$$

(i) If k ≤ 1, then T is called $\rho-$ nonexpansive

(ii) If k < 1, then T is called $\rho-$ contractive

LEMMA 2.12[2]: Let ρ ò $\Re$ and satisfy (UUC1). Let $\{t_n\}\subset(0,1)$ be bounded away from both 0 and 1. If there exists R > 0 such that

$$lim_{n\to\infty} sup\rho(f_n)\le R, lim_{n\to\infty} sup\rho(g_n)\le R \ and$$
$$lim_{n\to\infty}(t_n f_n+(1-t_n)\ g_n)=R,\ then$$
$$lim_{n\to\infty}\rho(f_n-g_n)=0$$

The sequence $\{t_n\}\subset(0,1)$ is said to be bounded away from 0 if there exists a > 0 such that $t_n\ge a$ for all n ∈ N. Similarly the sequence $\{t_n\}\subset(0,1)$ is said to be bounded away from 1 if there exists b < 1 such that $t_n\le b$ for all n ∈ N.

LEMMA 2.13[8]: Let T: C→ $P_\rho(C)$ be a multivalued mapping and

$$P_\rho^T(f)=\{g\in T:\rho(f-g)=d_\rho(f,\ Tf)\}$$

Then the following are equivalent:

i) $f\in F_\rho(T)$, that is $f\in T(f)$;

ii) $P_\rho^T(f)=\{f\}, i.e., f=g$ for each $g\in P_\rho^T(f)$;

iii) $f \in F_\rho(P_\rho^T(f))$ that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T)$ where $F(P_\rho^T)$ denotes the set of fixed points of P_ρ^T.

LEMMA 2.14: Let $\rho \in \Re$ and satisfy Δ_2 –condition $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then

$$lim_{n\to\infty}\rho(g_n) = 0 \ implies$$
$$lim_{n\to\infty}sup\,\rho(f_n + g_n) = lim_{n\to\infty}sup\,\rho(f_n)$$

and

$$lim_{n\to\infty}\rho(g_n) = 0 \ implies$$
$$lim_{n\to\infty}inf\,\rho(f_n + g_n) = lim_{n\to\infty}inf\,\rho(f_n).$$

LEMMA 2.15: Let $\rho \in \Re$ and A, B ∈ $P_\rho(L_\rho)$. For every f ∈ A, there exists g ∈ B such that

$$\rho(f - g) \le H_\rho(A, B)$$

MAIN RESULTS

Here, Picard-Ishikawa hybrid iterative scheme is as follows:

Let C$\subset L_\rho$ be a non empty $\rho-$bounded, closed and convex set and $T : C \to P_\rho(C)$ be a multivalued mappings. Let $f_1 \in C$ and { f_n }⊂ C be defined by

$$\begin{aligned} f_{n+1} &\in P_\rho^T(h_n) \\ g_n &= \beta_n u_n + (1-\beta_n) f_n \\ h_n &= \alpha_n v_n + (1-\alpha_n) f_n, \qquad n = 1, 2, \ldots \end{aligned} \tag{3.1}$$

where $u_n \in P_\rho^T(f_n), v_n \in P_\rho^T(g_n)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1) which are bounded away from both 0 and 1.

Before proving our main results, we firstly prove the following Lemma:

LEMMA 3.1: Let $\rho \in \Re$ and C ⊂ L_ρ be nonempty ρ-bounded and convex set. Suppose T: C $\to P_\rho$(C) be a multivalued mappings such that P_ρ^T is ρ-nonexpansive mapping with $F_\rho(T) \neq \phi$. Then the $lim_{n\to\infty}\rho(f_n - P_\rho^T(p))$ exists for all $p \in F_\rho(T)$.

PROOF. Let $p \in F_\rho(T)$ be arbitrary. Then by Lemma 2.15, we have

$$P_\rho^T(p) = \{p\}$$

Now from eq. (3.1), we have

$$\rho(f_{n+1} - p) \le H_\rho(P_\rho^T(h_n), P_\rho^T(p)) \le \rho(h_n - p) \tag{3.2}$$

Again from eq. (3.1), we get that

$$\rho(h_n - p) = \rho((1-\alpha_n) f_n + \alpha_n v_n - p)$$

$$\begin{aligned} &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(v_n - p) \\ &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \\ &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(g_n - p) \end{aligned} \tag{3.3}$$

$$\rho(g_n - p) = \rho(\beta_n u_n + (1-\beta_n) f_n - p)$$

$$\begin{aligned} &\le \beta_n \, \rho(u_n - p) + (1-\beta_n)\, \rho(f_n - p) \\ &\le \beta_n \, H_\rho\left(P_\rho^T(f_n), P_\rho^T(p)\right) + (1-\beta_n)\, \rho(f_n - p) \\ &\le \beta_n \rho(f_n - p) + (1-\beta_n)\, \rho(f_n - p) \\ &= \rho(f_n - p) \end{aligned} \tag{3.4}$$

Then from eq, (3.3) and eq. (3.4) in (3.2), we obtained that

$$\rho(f_{n+1} - p) \le \rho(f_n - p)$$

Therefore, the sequence { $\rho(f_n - p)$ } is decreasing. Hence $lim_{n\to\infty}\rho(f_n - p)$ exists for all $p \in F_\rho(T)$.

THEOREM 3.2: Let ρ ò $\Re$ satisfy (UUC1) and C ⊂ L_ρ be nonempty $\rho-$bounded, closed and convex set. Suppose T: C ⟶ P_ρ(C) be a multivalued mapping such that P_ρ^T is ρ-nonexpansive mapping with $F_\rho(T) \neq$ ϕ. Let $f_1 \in$C and { f_n } be given by (3.1). Then $lim_{n\to\infty}\rho(f_n - u_n) = 0$

PROOF. By Lemma 3.1, $lim_{n\to\infty}\rho(f_n - p)$ exists for all $p \in F_\rho(T)$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = R \tag{3.5}$$

From eq. (3.3) and eq. (3.4), we get

$$\lim_{n\to\infty} \sup \rho(g_n - p) \le R \tag{3.6}$$

This implies, $\rho(v_n - p) \le H_\rho\left(P_\rho^T(g_n), P_\rho^T(p)\right) \le \rho(g_n - p) \le \rho(f_n - p)$

[using eq. (3.4)]

which implies that

$$lim_{n\to\infty} \sup \rho(v_n - p) \le lim_{n\to\infty} \sup \rho(f_n - p)$$

or

$$lim_{n\to\infty} \sup \rho(v_n - p) \le R \tag{3.7}$$

Similarly

$$\rho(u_n - p) \le H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \le \rho(f_n - p)$$

$$lim_{n\to\infty} \sup \rho(u_n - p) \le lim_{n\to\infty} \sup \rho(f_n - p)$$

or $$\lim_{n\to\infty} \sup \rho\left(u_n - p\right) \le R \tag{3.8}$$

Since the sequence $\alpha_n \subset (0,1)$ is bounded away from 0 and 1, there exists $\alpha \in (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha$$

Now,

$$\rho\left(f_{n+1} - p\right) = \rho(\alpha_n v_n + (1-\alpha_n) f_n - p)$$

$$= \rho(\alpha_n \left(v_n - p\right) + (1-\alpha_n)\left(f_n - p\right))$$

$$\le \alpha_n \rho\left(v_n - p\right) + (1-\alpha_n)\, \rho\left(f_n - p\right)$$

$$\lim_{n\to\infty} \inf\ \rho\left(f_{n+1} - p\right) \le \lim_{n\to\infty} \inf\ (\alpha_n \rho\left(v_n - p\right) + (1-\alpha_n)\, \rho\left(f_n - p\right))$$

$$\le \lim_{n\to\infty} \inf\ (\alpha_n \rho\left(v_n - p\right)) + \lim_{n\to\infty} \inf\ (1-\alpha_n)\, \rho\left(f_n - p\right)$$

or $$R \le \alpha \lim_{n\to\infty} \inf\ \rho\left(v_n - p\right) + (1-\alpha)\, R$$

$$R \le \lim_{n\to\infty} \inf\ \rho(v_n - p) \tag{3.9}$$

From eq. (3.7) and eq. (3.9), we get

$$\lim_{n\to\infty} \rho\left(v_n - p\right) = R$$

Since $v_n \in P_\rho^T(g_n)$, then

$$\rho\left(v_n - p\right) \le \rho\left(g_n - p\right)$$

$$\lim_{n\to\infty} \inf \rho\left(g_n - p\right) \ge R \tag{3.10}$$

Then from eq. (3.6) and (3.10)

$$\lim_{n\to\infty} \rho\left(g_n - p\right) = R \tag{3.11}$$

Since the sequence $\{\beta_n\} \subset (0,\ 1)$ is bounded away from 0 and 1, there exists $\beta \in (0,\ 1)$ such that

$$\lim_{n\to\infty} \beta_n = \beta$$

Now,

$$\rho\left(g_n - p\right) = \rho(\beta_n u_n + (1-\beta_n) f_n - p)$$

$$\le \beta_n \rho\left(u_n - p\right) +\ (1-\beta_n)\, \rho\left(f_n - p\right)$$

which implies that

$$lim_{n\to\infty}inf\ \rho\left(g_n-p\right)\le lim_{n\to\infty}inf\,\beta_n\rho\left(u_n-p\right)+lim_{n\to\infty}inf\ (1-\beta_n)\ \rho\left(f_n-p\right)$$

$$R\le\beta lim_{n\to\infty}inf\ \rho\left(u_n-p\right)+(1-\beta)\ R$$

or $$lim_{n\to\infty}inf\ \rho\left(u_n-p\right)\ge R \qquad (3.12)$$

Then from eq. (3.8) and eq. (3.12), we have

$$lim_{n\to\infty}\rho\left(u_n-p\right)=R$$

Since, $\lim_{n\to\infty}\rho(g_n-p)=R$. This implies

$$lim_{n\to\infty}\rho(\beta_n u_n+(1-\beta_n)\,f_n-p)=R$$

or $$lim_{n\to\infty}\rho(\beta_n\left(u_n-p\right)+(1-\beta_n)\left(f_n-p\right))=R \qquad (3.13)$$

Then from eq. (3.5), eq. (3.8), eq. (3.13) and Lemma (2.12), we obtain that

$$lim_{n\to\infty}\rho\left(f_n-u_n\right)=0 \qquad (3.14)$$

or $$lim_{n\to\infty}dis_\rho\rho\left(f_n,P_\rho^T(f_n)\right)=0$$

THEOREM 3.3: Let $\rho\in\Re$ satisfy (UUC1) and C $\subset$ L_ρ be nonempty $\rho-$bounded and convex set. Let T: C $\to$ P_ρ(C) be a multivalued mapping such that P_ρ^T is $\rho-$nonexpansive mapping with $F_\rho(T)\neq\phi$. Let $f_1\in$ C and { f_n } be given by (3.1). Then the sequence { f_n } converges to a fixed point of T.

PROOF: Using the compactness of C, there must be a subsequence { f_{n_k} } of { f_n } and $f\in$ C such that $\rho\left(f_{n_k}-f\right)\to 0$ as $k\to\infty$. We will show that f is a fixed point of T, i.e., $f\in F_\rho(T)$.

Let $g\in P_\rho^T(f)$ be arbitrary. Then by Lemma 2.16, $g_k\in P_\rho^T(f_{n_k})$ such that

$$\rho\left(g_k-g\right)\le H_\rho(P_\rho^T(f_{n_k}),P_\rho^T(f))$$

$$\rho\left(\frac{f-g}{3}\right)=\rho\left(\frac{f-f_{n_k}}{3}+\frac{f_{n_k}-g_k}{3}+\frac{g_k-g}{3}\right)$$

$$\le\frac{1}{3}\rho(f-f_{n_k})+\frac{1}{3}\rho(f_{n_k}-g_k)+\frac{1}{3}\rho(g_k-g)$$

$$\le\rho(f-f_{n_k})+dis_\rho(f_{n_k}-P_\rho^T(f_{n_k}))+\rho(g_k-g)$$

$$\le\rho(f-f_{n_k})+dis_\rho(f_{n_k}-P_\rho^T(f_{n_k}))+H_\rho(P_\rho^T(f_{n_k}),P_\rho^T(f))$$

$$\le\rho(f-f_{n_k})+d_\rho(f_{n_k}-P_\rho^{T_1}(f_{n_k}))+\rho(f-f_{n_k})\to 0$$

as k$\to\infty$.

Hence, $f = g$ a.e. Since $g \in P_{\rho}^{T}(f)$ was arbitrary, we have $P_{\rho}^{T}(f) = \{f\}$. Thus by using Lemma 2.13, $f \in T(f)$ and thus $f \in F_{\rho}(T)$. This completes the proof.

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