Study of Equilibrium in Heterogeneous Strategies -A Triopoly Case

Bharti Kapoor*
Research Scholar IKG Punjab Technical University, Kapurthala, Punjab (India)
Dr.Ashish Arora
Head PG Department of Mathematics, IKG Punjab Technical University, Kapurthala, Punjab (India)

Abstract
Triopoly is a market, in which there are three types of competitors in the market. In this study, three types of Triopoly models with heterogeneous strategies are analyzed. Players using heterogeneous strategies is more realistic situation and the respective models are comparative complicated to solve than the models, with homogeneous strategies. First model is linear model with linear cost and linear demand functions, second model is with linear demand and nonlinear cost function and third model is also nonlinear triopoly model, where nonlinearity has been introduced through demand function. A system of dynamical equations has been formulated. Further, equilibrium points are calculated and their stability conditions are examined. Three different decisional mechanisms for each of three competitors are introduced. In present research, competitors use heterogeneous strategies for earning maximum possible profit, instead of using homogeneous strategies. discussion pertains to the Triopoly with three different heterogeneous players: boundedly rational player, adaptive player and naive player.

Key words: Triopoly, Equilibrium, Heterogeneous, cost functions, demand functions

Assumption

(i) Isoelastic demand function and Linear cost

(ii) First player is boundedly rational, second player is adaptive and third is naive.

(iii) Goods produced are homogeneous.

Linear Triopoly Model

The underlying assumption is that in Triopoly, players are dealing with homogeneous goods which are perfect substitutes. Let quantity supplied be x_i , where $i = 1,2,3.$ Inverse Demand function is given by $Y = a - bX$, where a and b are positive constants. $X = x_1 + x_2 + x_3$ is the total supply. Cost function is $C_i = c_i x_i$

So, Profit Function for the ith firm is

$$\pi_i = Yx_i - C_i$$
$$= x_i(a - bX) - c_i x_i \quad ,i = 1,2,3.$$

i.e.

$$\pi_1 = x_1\left[a - b(x_1 + x_2 + x_3)\right] - c_1 x_1$$
$$\pi_2 = x_2\left[a - b(x_1 + x_2 + x_3)\right] - c_2 x_2$$
$$\pi_3 = x_3\left[a - b(x_1 + x_2 + x_3)\right] - c_3 x_3$$

Each player wants to maximize his profit. So, in order to find profit maximizing quantity, the marginal profit is given

by:
$$\frac{\partial \pi_1}{\partial x_1} = a - 2bx_1 - bx_2 - bx_3 - c_1$$
$$\frac{\partial \pi_2}{\partial x_2} = a - bx_1 - 2bx_2 - bx_3 - c_2$$
$$\frac{\partial \pi_3}{\partial x_3} = a - bx_1 - bx_2 - 2bx_3 - c_3$$

For $\frac{\partial \pi_1}{\partial x_1} = 0, \frac{\partial \pi_2}{\partial x_2} = 0, \frac{\partial \pi_3}{\partial x_3} = 0$, equations are

$$a - c_1 - b(x_2 + x_3) - 2bx_1 = 0$$
$$a - c_2 - b(x_1 + x_3) - 2bx_2 = 0$$
$$a - c_3 - b(x_1 + x_2) - 2bx_3 = 0$$

Solving first equation gives

$$x_1 = \frac{a - c_1 - b(x_2 + x_3)}{2b} \qquad (1)$$

By using the concept of maxima minima, we find that profit is the maximum for this value of x_1. This is reaction function for the first firm. Similarly, reaction function for all the three firms is given by

$$x_2 = \frac{1}{2b}\left(a - b(x_3 + x_1) - c_2\right) \qquad (2)$$

and $$x_3 = \frac{1}{2b}\left(a - b(x_2 + x_1) - c_3\right) \qquad (3)$$

The general reaction function is

$$x_i = \frac{1}{2b}(a - b\sum_{\substack{j=1 \\ j \neq i}}^{3} x_j - c_i)$$

The first player is taken to be boundedly rational, second to be adaptive player and third to be naïve player. Denote by $x_i(t)$ and $x_i(t+1)$, the output of the player i at the time t and

$t+1$ respectively. The first player being boundedly rational makes his output decisions on the basis of the expected marginal profit. The dynamical equation of the first player is $x_1(t+1) = x_1(t) + \alpha\, x_1(t)\dfrac{\partial \pi_1}{\partial x_1(t)}$, $t = 0,1,2,3...$, where $\alpha > 0$ is the speed of adjustment, which symbolises the rate at which firms adjusts his output in the next period according to the change in marginal profit of previous period

$$i.e.\, x_1(t+1) = x_1(t) + \alpha\, x_1(t)(a - 2bx_1(t) - b(x_2(t) + x_3(t)) - c_1)\ \text{using}(1) \quad (4)$$

Again, the second being adaptive player calculates his output with the weights of his output of previous period and reaction function. The dynamical equation of the second player

is $x_2(t+1) = (1-\lambda)x_2(t) + \dfrac{\lambda}{2b}(a - b(x_1(t) + x_3(t)) - c_2)$

(5)

where $0 \le \lambda \le 1$ is the speed of adjustment

Also, the dynamical equation of the naive player is

$$x_3(t+1) = \frac{1}{2b}(a - b(x_1(t) + x_2(t)) - c_3) \quad (6)$$

These values of x_1, x_2 and x_3 obtained in equation (4), (5) and (6) represent the reaction functions of first, second and third firms respectively. Using these functions, firms can find the level of output to be produced in the time *'t+1'*.Level of output so obtained will give maximum profit.

Boundary, Nash Equilibrium Points and their Stability

The equilibrium point of the Triopoly game is obtained by the nonnegative fixed point of the system of nonlinear equations (4),(5) and (6).For finding fixed points it is needed to find $x_i(t+1) = x_i(t)$, $i = 1,2,3$ in each of (4), (5) and (6), So, system of equations is given by

$$x_1(a - 2bx_1 - b(x_2 + x_3) - c_1) = 0 \quad (7)$$
$$a - 2bx_2 - b(x_1 + x_3) - c_2 = 0 \quad (8)$$
$$a - 2bx_3 - b(x_1 + x_2) - c_3 = 0 \quad (9)$$

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ing equations (8) and (9),

$a - bx_1 = 2bx_2 + bx_3 + c_2$ and $a - bx_1 = 2bx_3 + bx_2 + c_3$

So, $2bx_3 + bx_2 + c_3 = 2bx_2 + bx_3 + c_2$ gives $x_2 = \dfrac{bx_3 + c_3 - c_2}{b}$.

Substituting this value in eq. (9) to get

$$bx_1 = a - 2bx_3 - bx_2 - c_3$$

$$\Rightarrow x_1 = \frac{1}{b}\left(a - 2bx_3 - \left(bx_3 + c_3 - c_2\right) - c_3\right)$$

$$\Rightarrow x_1 = \frac{a - 3bx_3 + c_2 - 2c_3}{b} \qquad (10)$$

Then from (7) Either $x_1 = 0$ *or* $a - 2bx_1 - b(x_2 + x_3) - c_1 = 0$

Using values of x_1 and x_2 obtained above in $a - 2bx_1 - b(x_2 + x_3) - c_1 = 0$ to get value of x_3

$$i.e.\ a - 2b\left(\frac{a - 3bx_3 + c_2 - 2c_3}{b}\right) - b\left(\frac{bx_3 + c_3 - c_2}{b} + x_3\right) - c_1 = 0$$

$$i.e.\ a - 2(a - 3bx_3 + c_2 - 2c_3) - (bx_3 + c_3 - c_2 + bx_3) - c_1 = 0$$

$$i.e.\ a - 2a + 6bx_3 - 2c_2 + 4c_3 - (bx_3 + c_3 - c_2 + bx_3) - c_1 = 0$$

$$i.e.\ 4bx_3 = a + c_2 + c_1 - 3c_3$$

So, value of $x_3 = \frac{a + c_2 + c_1 - 3c_3}{4b}$, By back substitution, values of x_1 and x_2 will be obtained, then values of x_1 and x_2 are given below:

$$x_2 = \frac{b\left(\frac{a + c_2 + c_1 - 3c_3}{4b}\right) + c_3 - c_2}{b}$$

$$i.e.\ x_2 = \frac{a + c_3 + c_1 - 3c_2}{4b}$$

and

$$x_1 = \frac{a - 3b\left(\frac{a + c_2 + c_1 - 3c_3}{4b}\right) + c_2 - 2c_3}{b}$$

$$i.e.\ x_1 = \frac{4a - 3(a + c_2 + c_1 - 3c_3) + 4c_2 - 8c_3}{4b}$$

$$i.e.\ x_1 = \frac{a + c_2 - 3c_1 + c_3}{4b}$$

Also, for $x_1 = 0$ equations (8) and (9) reduce to $(a-c_2)-2bx_2-bx_3=0$ and $(a-c_3)-bx_2-2bx_3=0$ respectively.

Multiplying first equation by '2' and subtracting from second gives

$$a-2c_2+c_3-3bx_2=0$$

$$i.e.x_2=\frac{a-2c_2+c_3}{3b} \tag{11}$$

Back substitution in first equation gives

$$(a-c_2)-2b\left(\frac{a-2c_2+c_3}{3b}\right)-bx_3=0$$
$$i.e.3(a-c_2)-2(a-2c_2+c_3)-3bx_3=0$$
$$i.e..x_3=\frac{a+c_2-2c_3}{3b} \tag{12}$$

So, For two values of x_1 , the two equilibrium points are:

$$E_1=\left(0,\frac{a-2c_2+c_3}{3b},\frac{a+c_2-2c_3}{3b}\right) \quad \text{and} \quad E_2=\left(\begin{array}{c}\frac{a+c_2+c_3-3c_1}{4b},\\ \frac{a+c_3+c_1-3c_2}{4b},\\ \frac{a+c_2+c_1-3c_3}{4b}\end{array}\right) \tag{13}$$

Where, E_1 is boundary equilibrium point and E_2 is Nash Equilibrium point. To check the stability of the equilibrium point, The Jacobian matrix of the system of equations given by (4),(5) and (6) at the equilibrium points is calculated first, then nature of Eigen values of this Jacobian matrix at the equilibrium points will determine the stability of equilibrium points. Jacobian matrix is given by

$$J=\begin{bmatrix}\frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3}\\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3}\\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3}\end{bmatrix}$$

$$i.e \quad J = \begin{bmatrix} 1+\alpha\left(a-4bx_1-b\left(x_2+x_3\right)-c_1\right) & -\alpha bx_1 & -\alpha bx_1 \\ -\frac{\lambda}{2} & 1-\lambda & -\frac{\lambda}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

At the boundary equilibrium point E_1, the Jacobian matrix is

$$J\left(E_1\right) = \begin{bmatrix} 1+\alpha\left[a-c_1-b\left(\frac{a-2c_2+c_3}{3b}+\frac{a+c_2-2c_3}{3b}\right)\right] & 0 & 0 \\ -\frac{\lambda}{2} & 1-\lambda & -\frac{\lambda}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\alpha\left[a-c_1-\frac{2a-c_2-c_3}{3}\right] & 0 & 0 \\ -\frac{\lambda}{2} & 1-\lambda & -\frac{\lambda}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\alpha\left[\frac{a-3c_1+c_2+c_3}{3}\right] & 0 & 0 \\ -\frac{\lambda}{2} & 1-\lambda & -\frac{\lambda}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Let β be the Eigen values of $J(E_1)$. Then Eigen values will be obtained if :

$$\begin{vmatrix} 1+\alpha\left(\frac{a-3c_1+e_2+c_3}{3}\right)-\beta & 0 & 0 \\ \frac{-\lambda}{2} & 1-\lambda-\beta & \frac{-\lambda}{2} \\ \frac{-1}{2} & \frac{-1}{2} & -\beta \end{vmatrix} = 0$$

$$\left\{1+\alpha\left(\frac{a-3c_1+c_2+c_3}{3}\right)-\beta\right\}\left\{\beta^2-\beta(1-\lambda)-\frac{\lambda}{4}\right\}=0$$

Eigen values of $J(E_1)$ are $\beta_1 = 1+\alpha\dfrac{a-3c_1+c_2+c_3}{3}$,

$\beta_{2,3} = \frac{1}{2}-\frac{1}{2}\lambda \pm \frac{1}{2}\sqrt{(1-2\lambda+\lambda^2+\lambda}$. As $0 \le \lambda \le 1$. So, $|\beta_1| > 1$ and $|\beta_{2,3}| < 1$

Then E_1 is saddle point of discrete dynamical system in (4),(5) and (6)

Similarly, at the Nash equilibrium point E_2,the Jacobian matrix is

$$J(E_2) = \begin{bmatrix} 1+\alpha\left(a-4bx_1^* - b\left(x_2^*+x_3^*\right)-c_1\right) & -\alpha bx_1^* & -\alpha bx_1^* \\ -\frac{\lambda}{2} & 1-\lambda & -\frac{\lambda}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}, \qquad (14)$$

$$x_1^* = \frac{a+c_2+c_3-3c_1}{4b}, \quad x_2^* = \frac{a+c_1+c_3-3c_2}{4b}, \quad x_3^* = \frac{a+c_2+c_1-3c_3}{4b}$$

Let γ be Eigen values of $J(E_2)$. Eigen values are obtained by taking:

$$\begin{vmatrix} (1+\alpha\left(a-4bx_1^* - b\left(x_2^*+x_3^*\right)-c_1\right)-\gamma & -\alpha bx_1^* & -\alpha bx_1^* \\ \frac{-\lambda}{2} & 1-\lambda-\gamma & \frac{-\lambda}{2} \\ \frac{-1}{2} & \frac{-1}{2} & -\gamma \end{vmatrix} = 0$$

$$\left(1+\alpha\left(a-4bx_1^* - b\left(x_2^*+x_3^*\right)-c_1\right)-\gamma\right)\left(-\gamma(1-\lambda-\gamma)-\frac{\lambda}{4}\right)+\alpha bx_1^*\left(\frac{\gamma\lambda}{2}-\frac{\lambda}{4}\right)-\alpha bx_1^*\left(\frac{\lambda}{2}+\frac{1-\lambda-\gamma}{2}\right)=0$$

$$\Rightarrow \left(1+\alpha\left(a-4bx_1^* - b\left(x_2^*+x_3^*\right)-c_1\right)-\gamma\right)\left(-\gamma+\lambda\gamma+\gamma^2-\frac{\lambda}{4}\right)+\alpha bx_1^*\left(\frac{\gamma\lambda}{2}-\frac{\lambda}{4}\right)-\alpha bx_1^*\left(\frac{\lambda}{2}+\frac{1-\lambda-\gamma}{2}\right)=0$$

Eigen values of the above Jacobian matrix are roots of characteristic

$$\gamma^3 + A_1\gamma^2 + A_2\gamma + A_3 = 0,$$

equation

$$A_1 = 2-\lambda+\alpha\left(a-4bx_1^*-b\left(x_2^*+x_3^*\right)-c_1\right), A_2 = -(1-\lambda)\left\{1+\alpha\left(a-4bx_1^*-bx_2^*-bx_3^*-c_1\right)\right\}+\alpha\frac{bx_1^*\lambda}{2}+\alpha\frac{bx_1^*}{2}+\frac{\lambda}{4}$$

$$A_3 = -\frac{\lambda}{4}\left[1+\alpha\left\{a-4bx_1^*-bx_2^*-bx_3^*-c_1\right\}\right]-\frac{\lambda\alpha bx_1^*}{4}-\frac{\alpha bx_1^*}{2} \quad (15)$$

Now Nash Equilibrium is asymptotically stable if all Eigen values given in equation has magnitude less than one. Which is possible if and only if

$$3+A_1-A_2-3A_3>0, \quad 1-A_2+A_3\left(A_1-A_3\right)>0 \quad and \quad 1-A_1+A_2-A_3>0 \quad (16)$$

Triopoly Model with Linear Demand and Non-Linear Cost Function

In this Triopoly model, nonlinearity is introduced through cost function and demand function is linear. With the above mentioned assumptions, quantity supplied is taken to be x_i , where $i=1,2,3$. Inverse Demand function is given by

$Y=a-bX$, where a and b are positive constants. $X=x_1+x_2+x_3$ is the total supply. Non-linear cost function is $C_i=c_ix_i^2$

So, Profit Function for the i^th^ firm is

$$\pi_i = Yx_i - C_i$$
$$= x_i(a-bX)-c_ix_i^2 \quad ,i=1,2,3.$$

Each player wants to maximize his profit. So, in order to find profit maximizing quantity, it is found that marginal profit

$$\frac{\partial\pi_1}{\partial x_1} = a-2bx_1-bx_2-bx_3-2c_1x_1 \quad (17)$$

$$For \ \frac{\partial\pi_i}{\partial x_1}=0$$

$$x_1 = \frac{1}{2(b+c_1)}\left(a-b(x_2+x_3)\right)$$

Also, $\frac{\partial\pi_2}{\partial x_2}=a-bx_1-2bx_2-bx_3-2c_2x_2$ gives

$$x_2 = \left(\frac{a-b\left(x_1+x_3\right)}{2\left(b+c_2\right)}\right)$$

And $\frac{\partial\pi_3}{\partial x_3}=a-bx_1-bx_2-2bx_3-2c_3x_3$ gives

$$x_3 = \left(\frac{a - b(x_1 + x_2)}{2(b + c_3)} \right)$$

Further investigation shows that for this value of x_1 , x_2 and x_3 profit is maximum. The general reaction function is

$$x_i = \frac{1}{2(b + c_i)}(a - b\sum_{\substack{j=1 \\ j \neq i}}^{3} x_j) \tag{18}$$

The first player is taken to be boundedly rational, second to be adaptive player and third to be naïve player. Denote by $x_i(t)$ and $x_i(t+1)$,the output of the player i at the time t and $t+1$ respectively. The first player being boundedly rational makes his output decisions on the basis of the expected marginal profit. The dynamical equation of the first player is $x_1(t+1) = x_1(t) + \alpha\, x_1(t) \frac{\partial \pi_1}{\partial x_1(t)}$, $t = 0,1,2,3...$, $where\ \alpha > 0$ is the speed of adjustment.

(19) $i.e. x_1(t+1) = x_1(t) + \alpha\, x_1(t)(a - 2(b + c_1)x_1 - b(x_2 + x_3))$ using(17)

(20)

Again, the second being adaptive player calculates his output with the weights of his output of previous period and reaction function. The dynamical equation of the second player is $x_2(t+1) = (1-\lambda)x_2(t) + \frac{\lambda}{2(b + c_2)}(a - b(x_1 + x_3))$

(21)

where $0 \le \lambda \le 1$ is the speed of adjustment .

Also, the dynamical equation of the naive player is

$$x_3(t+1) = \frac{1}{2(b + c_3)}(a - b(x_1 + x_2)) \tag{22}$$

Boundary , Nash Equilibrium Points and their Stability

Three equations (20),(21) and (22) collectively represent the discrete Dynamic system of triopoly game with heterogeneous competitors when cost function is nonlinear. The equilibrium point of the Triopoly game is obtained by the non-negative fixed point of the system of nonlinear equations (20),(21) and (22).Taking $x_i(t+1) = x_i(t)$, $t = 1,2,3$ in each of (20), (21) and (22),

$$x_1\left(a-2(b+c_1)x_1-b(x_2+x_3)\right)=0 \tag{23}$$
$$a-2(b+c_2)x_2-b(x_1+x_3)=0 \tag{24}$$
$$a-2(b+c_3)x_3-b(x_1+x_2)=0 \tag{25}$$

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rom(24) and (25)

$$a-bx_1=2(b+c_2)x_2+bx_3 \text{ and } a-bx_1=2(b+c_3)x_3+bx_2$$

So, $2(b+c_3)x_3+bx_2=2(b+c_2)x_2+bx_3$ gives $x_3=\frac{2c_2+b}{2c_3+b}x_2$. (26)

We substituting this value in eq. (24) to get

$$x_2\left[2(b+c_2)+b\frac{(2c_2+b)}{(2c_3+b)}\right]=a-bx_1$$

$$\Rightarrow x_2=\frac{(a-bx_1)(2c_3+b)}{2(b+c_2)(2c_3+b)+2bc_2+b^2}$$

Then from (26) $x_3=\frac{(a-bx_1)(2c_2+b)}{2(b+c_2)(2c_3+b)+2bc_2+b^2}$ (27)

From(23), Either $x_1=0$ *or* $a-2(b+c_1)x_1-b(x_2+x_3)=0$

This means either

$$x_1=0 \quad or \quad a-2(b+c_1)x_1-b\left(\frac{2ac_2+ab-2bc_2x_1-b^2x_1}{4bc_3+3b^2+4c_2c_3+4bc_2}+\frac{2ac_3+ab-2bc_3x_1-b^2x_1}{4bc_3+3b^2+4c_2c_3+4bc_2}\right)=0$$

i.e. either $x_1=0$ *or* $x_1\left[-2(b+c_1)+\frac{2b^3+2b^2(c_2+c_3)}{4bc_3+4c_2c_3+3b^2+4bc_2}\right]=-a+\frac{2ab(c_2+c_3)+2ab^2}{4bc_3+4c_2c_3+3b^2+4bc_2}$ *i.e.*

either $x_1=0$ *or* $x_1=\frac{a\left(b^2+2b(c_2+c_3)+4c_2c_3\right)}{2\left(2b^3+3b^2(c_1+c_2+c_3)+4b(c_1c_2+c_2c_3+c_1c_3)+4c_1c_2c_3\right)}$

For these two values of x_1 , we get two equilibrium points:

$$E_1 = \left(0, \frac{a(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}, \frac{a(b+2c_2)}{3b^2+4b(c_2+c_3)}\right) \quad \text{and}$$

$$E_2 = \left(\begin{array}{c} \frac{a(b^2+2b(c_2+c_3)+4c_2c_3)}{2(2b^3+3b^2(c_1+c_2+c_3)+4b(c_1c_2+c_2c_3+c_1c_3)+4c_1c_2c_3)}, \\ \frac{a(b^2+2b(c_1+c_3)+4c_1c_3)}{2(2b^3+3b^2(c_1+c_2+c_3)+4b(c_1c_2+c_2c_3+c_1c_3)+4c_1c_2c_3)}, \\ \frac{a(b^2+2b(c_1+c_2)+4c_1c_2)}{2(2b^3+3b^2(c_1+c_2+c_3)+4b(c_1c_2+c_2c_3+c_1c_3)+4c_1c_2c_3)} \end{array}\right) \quad (28)$$

Where, E_1 is boundary equilibrium point and E_2 is Nash Equilibrium point. To check the stability of the equilibrium point,the Jacobian matrix of the system of equations given by (20),(21) and (22) at the equilibrium points is first calculated, then nature of eigen values of this Jacobian matrix at the equilibrium points will determine the stability of equilibrium points. Jacobian matrix is given by

$$J = \begin{bmatrix} 1+\alpha(a-4(b+c_1)x_1-b(x_2+x_3)) & -\alpha b x_1 & -\alpha b x_1 \\ -\frac{\lambda b}{2(b+c_2)} & 1-\lambda & -\frac{\lambda b}{2(b+c_2)} \\ -\frac{b}{2(b+c_3)} & -\frac{b}{2(b+c_3)} & 0 \end{bmatrix} \quad (28)$$

At the boundary equilibrium point E_1, the Jacobian matrix is

$$J(E_1) = \begin{bmatrix} 1+\left(\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}\right) & 0 & 0 \\ -\frac{\lambda b}{2(b+c_2)} & 1-\lambda & -\frac{\lambda b}{2(b+c_2)} \\ -\frac{b}{2(b+c_3)} & -\frac{b}{2(b+c_3)} & 0 \end{bmatrix} \quad (29)$$

Let λ_1 be the Eigen value of $J(E_1)$, The Eigen values of the $J(E_1)$ are given by:

$$\begin{vmatrix} 1+\left(\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}\right)-\lambda_1 & 0 & 0 \\ -\frac{\lambda b}{2(b+c_2)} & 1-\lambda-\lambda_1 & -\frac{\lambda b}{2(b+c_2)} \\ -\frac{b}{2(b+c_3)} & -\frac{b}{2(b+c_3)} & -\lambda_1 \end{vmatrix} = 0$$

$$i.e.(1+\left(\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}\right)-\lambda_1)\left(-\lambda_1(1-\lambda-\lambda_1)-\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)=0$$

$$i.e.(1+\left(\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}\right)-\lambda_1)\left(-\lambda_1^2-\lambda_1(1-\lambda)+\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)=0$$

$$i.e.(1+\left(\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3}\right)-\lambda_1)\left(-\lambda_1^2-\lambda_1(1-\lambda)+\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)=0$$

$$\lambda_1=1+\frac{\alpha a(b+2c_2)(b+2c_3)}{3b^2+4b(c_2+c_3)+4c_2c_3} \text{ and } \lambda_{2,3}=\frac{1}{2}-\frac{1}{2}\lambda\pm\frac{1}{2}\sqrt{(1-2\lambda+\lambda^2+\frac{\lambda b^2}{(b+c_2)(b+c_3)})}\,.$$

As $0\le\lambda\le1$. So, $|\lambda_1|>1$ and $|\lambda_{2,3}|<1$. Then E_1 is saddle point of discrete dynamical system in (20),(21) and (22)

Similarly, at the Nash equilibrium point E_2, the Jacobian matrix is

$$J(E_2)=\begin{bmatrix} 1-2\alpha(b+c_1)x_1^* & -\alpha bx_1^* & -\alpha bx_1^* \\ -\frac{\lambda b}{2(b+c_2)} & 1-\lambda & -\frac{\lambda b}{2(b+c_2)} \\ -\frac{b}{2(b+c_3)} & -\frac{b}{2(b+c_3)} & 0 \end{bmatrix}, \tag{30}$$

$$x_1^*=\frac{a(b^2+2b(c_2+c_3)+4c_2c_3)}{2(2b^3+3b^2(c_1+c_2+c_3)+4b(c_1c_2+c_2c_3+c_1c_3)+4c_1c_2c_3)}$$

Eigen values x of $J(E_2)$ are given by:

$$\begin{vmatrix} 1-2\alpha(b+c_1)x_1^*-x & -\alpha bx_1^* & -\alpha bx_1^* \\ -\frac{\lambda b}{2(b+c_2)} & 1-\lambda-x & -\frac{\lambda b}{2(b+c_2)} \\ -\frac{b}{2(b+c_3)} & -\frac{b}{2(b+c_3)} & -x \end{vmatrix}=0$$

On solving above determinant, equation obtained is given by:

$$(1-2\alpha(b+c_1)x_1^*-x)\left(-x(1-\lambda)+x^2-\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)+$$

$$\alpha bx_1^*\left(\frac{\lambda bx}{2(b+c_2)}-\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)-\alpha bx_1^*\left(\frac{\lambda b^2}{4(b+c_2)(b+c_3)}+(1-\lambda-x)\frac{b}{2(b+c_3)}\right)=0$$

$$i.e. -x^3 + x^2\left(2-\lambda-2\alpha(b+c_1)x_1^*\right)+x\left(-(1-\lambda)(1-2\alpha(b+c_1)x_1^*)+\frac{\lambda\alpha b^2 x_1^*}{2(b+c_2)}+\frac{\alpha b^2 x_1^*}{2(b+c_3)}+\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)+$$
$$-\left((1-2\alpha(b+c_1)x_1^*)\frac{\lambda b^2}{4(b+c_2)(b+c_3)}\right)-\frac{\alpha x_1^*\lambda b^3}{2(b+c_2)(b+c_3)}-(1-\lambda)\frac{\alpha b^2 x_1^*}{2(b+c_3)}=0$$

Eigen values of the above Jacobian matrix are roots of characteristic equation

$$x^3 + A_1x^2 + A_2x + A_3 = 0,$$ where

$$A_1 = 2\alpha x_1^*(b+c_1)-2+\lambda,\ A_2 = -\alpha b x_1^*\left(\frac{\lambda b}{2(b+c_2)}+\frac{b}{2(b+c_3)}\right)-2\alpha x_1^*(c_1+b)+2\alpha x_1^*\lambda(c_1+b)-\lambda+1-\frac{\lambda b^2}{4(b+c_2)(b+c_3)}$$

$$A_3 = \frac{\lambda b^2}{4(b+c_2)(b+c_3)}(1-2\alpha c_1 x_1^*)+\frac{\alpha b^2 x_1^*(1-\lambda)}{2(b+c_3)}+\frac{\alpha x_1^*\lambda b^3}{2(b+c_2)(b+c_3)}$$

Now Nash Equilibrium is asymptotically stable if all Eigen values given in eq.(32) has magnitude less than one. Which is possible if f

$$3+A_1-A_2-3A_3>0,\qquad 1-A_2+A_3(A_1-A_3)>0 \qquad and \qquad 1-A_1+A_2-A_3>0 \tag{31}$$

Triopoly Model with Non-Linear Demand and Linear Cost Function

In this model, assumptions are same as mentioned above and quantity supplied be x_i, where $i=1,2,3$. $X = x_1+x_2+x_3$ is the total supply of goods in the market. Iso-elasic inverse demand function is given by $Y=\frac{1}{X}$. Here, cost function $C_i = c_i x_i \; i=1,2,3$ is linear. So, Profit Function for the 1st, 2nd and 3rd firms are:

$$\pi_1 = Yx_1 - C_1$$
$$= \frac{x_1}{(x_1+x_2+x_3)} - c_1x_1$$

$$\pi_2 = Yx_2 - C_2$$
$$= \frac{x_2}{(x_1+x_2+x_3)} - c_2x_2$$

$$\pi_3 = Yx_3 - C_3$$
$$= \frac{x_3}{(x_1+x_2+x_3)} - c_3x_3$$

As mentioned above , in order to find profit maximizing level of output, the marginal profit and value of output is found for which

$$\frac{\partial \pi_i}{\partial x_i} = 0$$

$$\frac{\partial \pi_1}{\partial x_1} = 0$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 - x_1}{(x_1 + x_2 + x_3)^2} - c_1 = 0$$

$$\Rightarrow \frac{x_2 + x_3}{(x_1 + x_2 + x_3)^2} - c_1 = 0$$

$$\Rightarrow \frac{(x_1 + x_2 + x_3)^2}{x_2 + x_3} = \frac{1}{c_1}$$

$$\Rightarrow x_1 = \frac{\sqrt{x_2 + x_3}}{\sqrt{c_1}} - x_2 - x_3$$

Also

$$\frac{\partial \pi_2}{\partial x_2} = 0$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 - x_2}{(x_1 + x_2 + x_3)^2} - c_2 = 0$$

$$\Rightarrow \frac{x_1 + x_3}{(x_1 + x_2 + x_3)^2} - c_2 = 0$$

$$\Rightarrow \frac{(x_1 + x_2 + x_3)^2}{x_1 + x_3} = \frac{1}{c_2}$$

$$\Rightarrow x_2 = \frac{\sqrt{x_1 + x_3}}{\sqrt{c_2}} - x_1 - x_3$$

And

$$\frac{\partial \pi_3}{\partial x_3} = 0$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 - x_3}{(x_1 + x_2 + x_3)^2} - c_3 = 0$$

$$\Rightarrow \frac{x_1 + x_2}{(x_1 + x_2 + x_3)^2} - c_3 = 0$$

$$\Rightarrow \frac{(x_1 + x_2 + x_3)^2}{x_1 + x_2} = \frac{1}{c_3}$$

$$\Rightarrow x_3 = \frac{\sqrt{x_1 + x_2}}{\sqrt{c_3}} - x_1 - x_2$$

As assumed in above model, first player is boundedly rational, second adaptive player and third naïve player, So, their output at time *(t+1)* is given below $x_1(t+1) = x_1(t) + \alpha\, x_1(t)\frac{\partial \pi_1}{\partial x_1(t)}$ $, t = 0,1,2,3...$, where $\alpha > 0$ is the speed of adjustment.

$$x_1(t+1) = x_1(t) + \alpha x_1(t)\left[\frac{x_2 + x_3}{(x_1 + x_2 + x_3)^2} - c_1\right]$$

(32) $x_2(t+1) = (1-\lambda)x_2(t) + \lambda\left[\sqrt{\frac{x_1(t)+x_3(t)}{c_2}} - x_1(t) - x_3(t)\right]$ where $0 \le \lambda \le 1$ is the speed of adjustment. (33)

For $\lambda = 1$, adaptive player becomes naïve. And

$$x_3(t+1) = \sqrt{\frac{x_1(t)+x_2(t)}{c_3}} - x_1(t) - x_2(t) \quad (34)$$

Equations (33),(34) and (35) collectively represents the three dimensional discrete dynamical system of the firms and with the help of these equations, a firm can determine the level of output to be produced in time $(t+1)$, using the value of output produced at time $'t'$ by itself and other two competing firms.

Conclusion: The present study having the assumption that first player is boundedly rational, second player is adaptive and third is naive. Goods produced are homogeneous. The first player being boundedly rational makes his output decidions on the basis of the expected marginal profits. Dynamical equations of all the players were found at time *t+1*.The boundary points E_1,and Nash equilibrium point E_2 were analyzed. E_2 is asymptotically stable if all eigen values has magnitude less than one, which is possible under some conditions. In linear demand and non-linear cost function E_1& E_2were calculated where E_1 become saddle point. Similarly $E_2$is again asymptotically stable under same conditions . Trioploy model with non linear demand and linear cost function is established and it is interesting to see their boundary points and Nash Equilibrium points in future.

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