Triple Positive Solutions for a Nonlinear $3n^{th}$ Order Three-Point Boundary Value Problem on Time Scales

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Abstract

In this paper, we establish the existence of at least three positive solutions for $3n^{th}$ order three-point boundary value problem on time scales by using Avery generalization of the Leggett–Williams fixed point theorem.

Key words: Green's function, boundary value problem, time scale, positive solution, cone, fixed point theorem.

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1 Introduction

The theory of time scales [5, 6] was initiated by Hilger [11] in his Ph.D. thesis in 1988. This theory unifies not only continuous and discrete theory, but also provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. This theory is widely applied to various real life situations like epidemic models, stock market, mathematical modeling of physical and biological systems and certain economically important phenomena contain processes that feature elements of both continuous and discrete.

The existence of positive solutions of the higher order boundary value problems (BVPs) on time scales have been studied extensively due to their striking applications to almost all area of science, engineering and technology. To mention a few papers along these lines are Henderson [10], Chyan [9], Anderson et.al [1, 2, 3], Cetin and Topal [7, 8], Hu and Zhou [12], Prasad and Sreedhar [13, 14, 15] and Yaslan [16, 17].

In this paper, we are concerned with the existence of positive solutions for $3n^{th}$ order BVP on time scales,

$$(-1)^n y^{\Delta^{(3n)}}(t) = f(y(t), y^{\Delta^{(3)}}(t), y^{\Delta^{(6)}}(t), ..., y^{\Delta^{(3n-3)}}(t)), \quad t \in [t_1, \sigma(t_3)] \tag{1.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned} \alpha_{3i-2,1}y^{\Delta^{(3i-3)}}(t_1) + \alpha_{3i-2,2}y^{\Delta^{(3i-2)}}(t_1) + \alpha_{3i-2,3}y^{\Delta^{(3i-1)}}(t_1) = 0,\\ \alpha_{3i-1,1}y^{\Delta^{(3i-3)}}(t_2) + \alpha_{3i-1,2}y^{\Delta^{(3i-2)}}(t_2) + \alpha_{3i-1,3}y^{\Delta^{(3i-1)}}(t_2) = 0,\\ \alpha_{3i,1}y^{\Delta^{(3i-3)}}(\sigma(t_3)) + \alpha_{3i,2}y^{\Delta^{(3i-2)}}(\sigma(t_3)) + \alpha_{3i,3}y^{\Delta^{(3i-1)}}(\sigma(t_3)) = 0, \end{aligned}\right\} \tag{1.2}$$

for $1 \leq i \leq n$, where $n \geq 1$, $\alpha_{3i-2,j}, \alpha_{3i-1,j}, \alpha_{3i,j}$, for $j = 1, 2, 3$, are real constants, $t_1 < t_2 < \sigma(t_3)$ and $f : \mathbb{R}^{+^n} \to \mathbb{R}^+$ is continuous.

For convenience, we use the following notations. For $1 \leq i \leq n$, let us denote $\beta_{i_j} = \alpha_{3i-3+j,1}t_j + \alpha_{3i-3+j,2}$, $\gamma_{i_j} = \alpha_{3i-3+j,1}t_j^2 + \alpha_{3i-3+j,2}(t_j + \sigma(t_j)) + 2\alpha_{3i-3+j,3}$, where $j = 1, 2$; $\beta_{i_3} = \alpha_{3i,1}\sigma(t_3) + \alpha_{3i,2}$ and $\gamma_{i_3} = \alpha_{3i,1}(\sigma(t_3))^2 + \alpha_{3i,2}(\sigma(t_3) + \sigma^2(t_3)) + 2\alpha_{3i,3}$. Also, for $1 \leq i \leq n$, we define

$$m_{i_{jk}} = \frac{\alpha_{3i-3+j,1}\gamma_{i_k} - \alpha_{3i-3+k,1}\gamma_{i_j}}{2(\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j})}, \quad M_{i_{jk}} = \frac{\beta_{i_j}\gamma_{i_k} - \beta_{i_k}\gamma_{i_j}}{\alpha_{3i-3+j,1}\beta_{i_k} - \alpha_{3i-3+k,1}\beta_{i_j}},$$

where $j, k = 1, 2, 3$ and let $p_i = \max\{m_{i_{12}}, m_{i_{13}}, m_{i_{23}}\}$,
$q_i = \min\left\{m_{i_{23}} + \sqrt{m_{i_{23}}^2 - M_{i_{23}}},\ \ m_{i_{13}} + \sqrt{m_{i_{13}}^2 - M_{i_{13}}}\right\}$,
$d_i = \alpha_{3i-2,1}(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) - \beta_{i_1}(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) + \gamma_{i_1}(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})$ and $l_{i_j} = \alpha_{3i-3+j,1}\sigma(s)\sigma^2(s) - \beta_{i_j}(\sigma(s) + \sigma^2(s)) + \gamma_{i_j}$, where $j = 1, 2, 3$. We assume the following conditions throughout this paper:

(A1) $\alpha_{3i-2,1} > 0, \alpha_{3i-1,1} > 0, \alpha_{3i,1} > 0$ and $\frac{\alpha_{3i,2}}{\alpha_{3i,1}} > \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} > \frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}}$, for all $1 \leq i \leq n$,

(A2) $p_i \leq t_1 < t_2 < \sigma(t_3) \leq q_i$ and $2\alpha_{3i-2,3}\alpha_{3i-2,1} > \alpha_{3i-2,2}^2$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} < \alpha_{3i-1,2}^2$, $2\alpha_{3i,3}\alpha_{3i,1} > \alpha_{3i,2}^2$, for all $1 \leq i \leq n$,

(A3) $m_{i_{23}}^2 > M_{i_{23}}$, $m_{i_{12}}^2 < M_{i_{12}}$, $m_{i_{13}}^2 > M_{i_{13}}$ and $d_i > 0$, for all $1 \leq i \leq n$,

(A4) The point $t \in [t_1, \sigma(t_3)]$ is not left dense and right scattered at the same time.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. In Section 3, we establish a criteria for the existence of at least three positive solutions for the BVP (1.1)-(1.2) by using Avery generalization of the Leggett–Williams fixed point theorem.

2 Green's function and bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function.

For $1 \le i \le n$, let $G_i(t,s)$ be the Green's function for the homogeneous BVP,

$$-y^{\Delta^3}(t) = 0, \qquad t \in [t_1, \sigma(t_3)], \tag{2.1}$$

satisfying the general three-point boundary conditions,

$$\left.\begin{aligned} \alpha_{3i-2,1}y(t_1) + \alpha_{3i-2,2}y^{\Delta}(t_1) + \alpha_{3i-2,3}y^{\Delta^2}(t_1) = 0,\\ \alpha_{3i-1,1}y(t_2) + \alpha_{3i-1,2}y^{\Delta}(t_2) + \alpha_{3i-1,3}y^{\Delta^2}(t_2) = 0,\\ \alpha_{3i,1}y(\sigma(t_3)) + \alpha_{3i,2}y^{\Delta}(\sigma(t_3)) + \alpha_{3i,3}y^{\Delta^2}(\sigma(t_3)) = 0. \end{aligned}\right\} \tag{2.2}$$

Lemma 2.1 *For $1 \le i \le n$, the Green's function $G_i(t,s)$ for the homogeneous BVP (2.1)-(2.2) is given by*

$$G_i(t,s) = \begin{cases} \underset{t\in[t_1,t_2]}{G_i(t,s)} = \begin{cases} G_{i_1}(t,s), & t_1 < \sigma(s) < t \le t_2 < \sigma(t_3)\\ G_{i_2}(t,s), & t_1 \le t < s < t_2 < \sigma(t_3)\\ G_{i_3}(t,s), & t_1 \le t < t_2 < s < \sigma(t_3) \end{cases}\\[2ex] \underset{t\in[t_2,\sigma(t_3)]}{G_i(t,s)} = \begin{cases} G_{i_4}(t,s), & t_1 < t_2 < \sigma(s) < t \le \sigma(t_3)\\ G_{i_5}(t,s), & t_1 < t_2 \le t < s < \sigma(t_3)\\ G_{i_6}(t,s), & t_1 \le \sigma(s) < t_2 < t < \sigma(t_3) \end{cases} \end{cases} \tag{2.3}$$

where

$$\begin{aligned} G_{i_1}(t,s) =& \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} -\\ & \alpha_{3i,1}\beta_{i_2})]l_{i_1},\\ G_{i_2}(t,s) =& \frac{1}{2d_i}\{[-(\beta_{i_1}\gamma_{i_3} - \beta_{i_3}\gamma_{i_1}) + t(\alpha_{3i-2,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_1}) - t^2(\alpha_{3i-2,1}\beta_{i_3} -\\ & \alpha_{3i,1}\beta_{i_1})]l_{i_2} + [(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) +\\ & t^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]l_{i_3}\}, \end{aligned}$$

$$G_{i_3}(t,s) = \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + t^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]l_{i_3},$$

$$G_{i_4}(t,s) = \frac{1}{2d_i}\{[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]l_{i_1} + [(\beta_{i_1}\gamma_{i_3} - \beta_{i_3}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_1}) + t^2(\alpha_{3i-2,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_1})]l_{i_2}\},$$

$$G_{i_5}(t,s) = \frac{1}{2d_i}[(\beta_{i_1}\gamma_{i_2} - \beta_{i_2}\gamma_{i_1}) - t(\alpha_{3i-2,1}\gamma_{i_2} - \alpha_{3i-1,1}\gamma_{i_1}) + t^2(\alpha_{3i-2,1}\beta_{i_2} - \alpha_{3i-1,1}\beta_{i_1})]l_{i_3},$$

$$G_{i_6}(t,s) = \frac{1}{2d_i}[-(\beta_{i_2}\gamma_{i_3} - \beta_{i_3}\gamma_{i_2}) + t(\alpha_{3i-1,1}\gamma_{i_3} - \alpha_{3i,1}\gamma_{i_2}) - t^2(\alpha_{3i-1,1}\beta_{i_3} - \alpha_{3i,1}\beta_{i_2})]l_{i_1}.$$

Lemma 2.2 *Assume that the conditions (A1)-(A4) are satisfied. Then, for* $1 \le i \le n$, *the Green's function* $G_i(t,s)$ *satisfies the following inequality,*

$$m_i G_i(\sigma(s),s) \le G_i(t,s) \le G_i(\sigma(s),s), \text{ for all } (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3], \tag{2.4}$$

where

$$0 < m_i = \min\left\{ \frac{G_{i_1}(\sigma(t_3),s)}{G_{i_1}(t_1,s)}, \frac{G_{i_3}(t_1,s)}{G_{i_3}(\sigma(t_3),s)}, \frac{G_{i_2}(t_1,s)}{G_{i_2}(\sigma(t_3),s)}, \frac{G_{i_4}(\sigma(t_3),s)}{G_{i_4}(t_1,s)} \right\} < 1.$$

Lemma 2.3 *Assume that the conditions (A1)-(A4) are satisfied and* $G_i(t,s)$ *as in (2.3). Let us take* $H_1(t,s) = G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_{t_1}^{\sigma(t_3)} H_{j-1}(t,r) G_j(r,s) \Delta r,$$

for $2 \le j \le n$, *then* $H_n(t,s)$ *is the Green's function for the homogeneous BVP corresponding to (1.1)-(1.2).*

Lemma 2.4 *Assume that the conditions (A1)-(A4) holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j \quad \text{and} \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in Lemma 2.3 satisfies*

$$0 \le H_n(t,s) \le K\|G_n(.,s)\|, \text{ for all } (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3]$$

and

$$H_n(t,s) \geq m_n L\|G_n(.,s)\|, \text{ for all } (t,s) \in [t_2, \sigma(t_3)] \times [t_1, t_3],$$

where m_n *is given as in Lemma 2.2,*

$$K_j = \int_{t_1}^{\sigma(t_3)} \|G_j(.,s)\|\Delta s > 0, \text{ for } 1 \leq j \leq n,$$

$$L_j = \int_{t_2}^{\sigma(t_3)} \|G_j(.,s)\|\Delta s > 0, \text{ for } 1 \leq j \leq n$$

and $\|.\|$ *is defined by*

$$\|x\| = \max_{t \in [t_1, \sigma(t_3)]} |x(t)|.$$

Let $D = \{v|v : C[t_1, \sigma(t_3)]\}$. For each $1 \leq j \leq n-1$, define the operator $T_j : D \to D$ by

$$T_j v(t) = \int_{t_1}^{\sigma(t_3)} H_j(t,s) v(s) \Delta s, \quad t \in [t_1, \sigma(t_3)]$$

and these integrals are converges. By the construction of T_j and the properties of $H_j(t,s)$, it is clear that

$$(-1)^j (T_j v)^{(\Delta\nabla)^j}(t) = v(t), \quad t \in [t_1, \sigma(t_3)],$$

$$\alpha_{3i-2,1}(T_j v)^{\Delta^{(3i-3)}}(t_1) + \alpha_{3i-2,2}(T_j v)^{\Delta^{(3i-2)}}(t_1) + \alpha_{3i-2,3}(T_j v)^{\Delta^{(3i-1)}}(t_1) = 0,$$
$$\alpha_{3i-1,1}(T_j v)^{\Delta^{(3i-3)}}(t_2) + \alpha_{3i-1,2}(T_j v)^{\Delta^{(3i-2)}}(t_2) + \alpha_{3i-1,3}(T_j v)^{\Delta^{(3i-1)}}(t_2) = 0,$$
$$\alpha_{3i,1}(T_j v)^{\Delta^{(3i-3)}}(\sigma(t_3)) + \alpha_{3i,2}(T_j v)^{\Delta^{(3i-2)}}(\sigma(t_3)) + \alpha_{3i,3}(T_j v)^{\Delta^{(3i-1)}}(\sigma(t_3)) = 0,$$

for $1 \leq i \leq j$. Hence, we see that the BVP (1.1)-(1.2) has a solution if and only if the following BVP has a solution,

$$v^{\Delta^3}(t) + f(T_{n-1}v(t), T_{n-2}v(t), ..., T_1 v(t), v(t)) = 0, \quad t \in [t_1, \sigma(t_3)] \tag{2.5}$$

$$\left.\begin{aligned} \alpha_{3i-2,1}v(t_1) + \alpha_{3i-2,2}v^{\Delta}(t_1) + \alpha_{3i-2,3}v^{\Delta^2}(t_1) = 0, \\ \alpha_{3i-1,1}v(t_2) + \alpha_{3i-1,2}v^{\Delta}(t_2) + \alpha_{3i-1,3}v^{\Delta^2}(t_2) = 0, \\ \alpha_{3i,1}v(\sigma(t_3)) + \alpha_{3i,2}v^{\Delta}(\sigma(t_3)) + \alpha_{3i,3}v^{\Delta^2}(\sigma(t_3)) = 0, \end{aligned}\right\} \tag{2.6}$$

for $i = n-1$. Indeed, if y is a solution of the BVP (1.1)-(1.2), then $v(t) = y^{\Delta^{(3n-3)}}(t)$ is a solution of the BVP (2.5)-(2.6). Conversely, if v is a solution of the BVP (2.5)-(2.6), then $y(t) = T_{n-1}v(t)$ is a solution of the BVP (1.1)-(1.2). In fact, $y(t)$ is represented as

$$y(t) = \int_{t_1}^{\sigma(t_3)} H_{n-1}(t,s)v(s)\Delta s,$$

where

$$v(s) = \int_{t_1}^{\sigma(t_3)} G_n(s,\tau)f(T_{n-1}v(\tau), T_{n-2}v(\tau), ..., T_1v(\tau), v(\tau))\Delta\tau.$$

3 Triple positive solutions

In this section, we establish the existence of at least three positive solutions for the BVP (1.1)-(1.2), by using Avery generalization of the Leggett–Williams fixed point theorem.

Let B be a real Banach space with cone P. A map $\alpha : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P if α is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Similarly, we say that a map $\beta : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \leq \lambda\beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let γ, β, θ be nonnegative continuous convex functional on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', we define the following convex sets

$$P(\gamma, c') = \{y \in P | \gamma(y) < c'\},$$

$$P(\gamma, \alpha, a', c') = \{y \in P | a' \leq \alpha(y),\ \gamma(y) \leq c'\},$$

$$Q(\gamma, \beta, d', c') = \{y \in P | \beta(y) \leq d',\ \gamma(y) \leq c'\},$$

$$P(\gamma, \theta, \alpha, a', b', c') = \{y \in P | a' \leq \alpha(y),\ \theta(y) \leq b',\ \gamma(y) \leq c'\},$$

$$Q(\gamma, \beta, \psi, h', d', c') = \{y \in P | h' \leq \psi(y),\ \beta(y) \leq d',\ \gamma(y) \leq c'\}.$$

In obtaining multiple positive solutions of the BVP (1.1)-(1.2), the following Avery generalization of the Leggett–Williams fixed point theorem, so called Five Functionals Fixed Point Theorem will be fundamental.

Theorem 3.1 [4] *Let P be a cone in a real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β and θ are nonnegative continuous convex functionals on P such that, for some positive numbers c' and k,*

$$\alpha(y) \leq \beta(y) \quad \text{and} \quad \| y \| \leq k\gamma(y), \quad \text{for all} \quad y \in \overline{P(\gamma, c')}.$$

Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants $h', d', a', b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied.

$(B1)$ $\{y \in P(\gamma, \theta, \alpha, a', b', c') | \alpha(y) > a'\} \neq \emptyset$ *and* $\alpha(Ty) > a'$, *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,

$(B2)$ $\{y \in Q(\gamma, \beta, \psi, h', d', c') | \beta(y) < d'\} \neq \emptyset$ *and* $\beta(Ty) < d'$, *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,

$(B3)$ $\alpha(Ty) > a'$, *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,

$(B4)$ $\beta(Ty) < d'$, *provided* $y \in Q(\gamma, \beta, d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that

$$\beta(y_1) < d', \quad a' < \alpha(y_2) \text{ and } d' < \beta(y_3) \quad \text{with} \quad \alpha(y_3) < a'.$$

Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j} \tag{3.1}$$

Let $B = \{v | v : C[t_1, \sigma(t_3)]\}$ be the Banach space equipped with the norm

$$\|v\| = \max_{t \in [t_1, \sigma(t_3)]} |v(t)|.$$

Define the cone $P \subset B$ by

$$P = \left\{ \ v \in B : v(t) \geq 0 \text{ on } [t_1, \sigma(t_3)] \text{ and } \min_{t \in [t_2, \sigma(t_3)]} v(t) \geq M\|v\| \ \right\},$$

where M is given as in (3.1). Now, let $I_1 = \left[\frac{t_2+t_3}{3}, \sigma(t_3)\right]$ and define the nonnegative continuous concave functionals $\alpha, \ \psi$ and the nonnegative continuous convex functionals $\beta, \ \theta, \ \gamma$ on P by

$$\gamma(v) = \max_{t \in [t_1, \sigma(t_3)]} |v(t)|, \quad \psi(v) = \min_{t \in I_1} |v(t)|, \quad \beta(v) = \max_{t \in I_1} |v(t)|,$$
$$\alpha(v) = \min_{t \in [t_2, \sigma(t_3)]} |v(t)| \quad \text{and} \quad \theta(v) = \max_{t \in [t_2, \sigma(t_3)]} |v(t)|.$$

We observe that for any $v \in P$,

$$\alpha(v) = \min_{t \in [t_2, \sigma(t_3)]} |v(t)| \leq \max_{t \in I_1} |v(t)| = \beta(v) \tag{3.2}$$

and

$$\|v\| \leq \frac{1}{M} \min_{t\in[t_2,\sigma(t_3)]} v(t) \leq \frac{1}{M} \max_{t\in[t_1,\sigma(t_3)]} |v(t)| = \frac{1}{M}\gamma(v). \tag{3.3}$$

We are now ready to present the main result of this section. We denote

$$M_j = \int_{s\in I_1} \|G_j(.,s)\|\Delta s, \text{ for } 1 \leq j \leq n.$$

Theorem 3.2 *Suppose there exist* $0 < a' < b' < \frac{b'}{M} \leq c'$ *such that* f *satisfies the following conditions:*

$(A1)$ $f(u_{n-1}, u_{n-2}, ..., u_1, u_0) < \frac{a'}{K_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, ..., |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j LMa'M_j, \frac{c'KK_j}{M}] \times [Ma', a']$,

$(A2)$ $f(u_{n-1}, u_{n-2}, ..., u_1, u_0) > \frac{b'}{MK_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, ..., |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j Lb'L_j, \frac{c'KK_j}{M}] \times [b', \frac{b'}{M}]$,

$(A3)$ $f(u_{n-1}, u_{n-2}, ..., u_1, u_0) < \frac{c'}{K_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, ..., |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[0, \frac{c'KK_j}{M}] \times [0, c']$.

Then the BVP (1.1)-(1.2) *has at least three positive solutions.*

Proof: Define the operator $T : P \to B$ by

$$Tv(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s. \tag{3.4}$$

It is obvious that a fixed point of T is the solution of the BVP (2.5)-(2.6). We seek three fixed points $v_1, v_2, v_3 \in P$ of T. First, we show that $T : P \to P$. Let $v \in P$. Clearly, $Tv(t) \geq 0$, for $t \in [t_1, \sigma(t_3)]$. Also, noting that Tv satisfies the boundary conditions (2.6). Then, we have

$$\begin{aligned}
\min_{t\in[t_2,\sigma(t_3)]} Tv(t) =& \min_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\
&\geq M \int_{t_1}^{\sigma(t_3)} G_n(s,s) f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\
&\geq M \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\
&\geq M\|Tv\|.
\end{aligned}$$

Hence, $Tv \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (3.2) and (3.3), for each $v \in P$, we have $\alpha(v) \leq \beta(v)$ and $\|v\| \leq \frac{1}{M}\gamma(v)$.

To show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $v \in \overline{P(\gamma, c')}$. This implies $\|v\| \leq \frac{c'}{M}$. Using Lemma 2.4, for $1 \leq j \leq n-1$ and $t \in [t_1, \sigma(t_3)]$, we have

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{\sigma(t_3)} H_j(t,s) v(s) \Delta s \\ &\leq \frac{c'}{M} \int_{t_1}^{\sigma(t_3)} H_j(t,s) \Delta s \\ &\leq \frac{c'}{M} K \int_{t_1}^{\sigma(t_3)} \|G_j(.,s)\| \Delta s = \frac{c' K K_j}{M}. \end{aligned}$$

We may now use condition $(A3)$ to obtain

$$\begin{aligned} \gamma(Tv) &= \max_{t \in [t_1, \sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1 v(s), v(s)) \Delta s \\ &\leq \frac{c'}{K_n} \int_{t_1}^{\sigma(t_3)} G_n(s,s) \Delta s = c'. \end{aligned}$$

Therefore, $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$.

We first verify that conditions $(B1), (B2)$ of Theorem 3.1 are satisfied. It is obvious that

$$\{v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c') | \alpha(v) > b'\} \neq \emptyset$$

and

$$\{v \in Q(\gamma, \beta, \psi, Ma', a', c') | \beta(v) < a'\} \neq \emptyset.$$

Next, let $v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c')$ or $v \in Q(\gamma, \beta, \psi, Ma', a', c')$.
Then, for $1 \leq j \leq n-1$,

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{\sigma(t_3)} H_j(t,s) v(s) \Delta s \\ &\leq \frac{c'}{M} \int_{t_1}^{\sigma(t_3)} H_j(t,s) \Delta s \\ &\leq \frac{c'}{M} K \int_{t_1}^{\sigma(t_3)} \|G_j(.,s)\| \Delta s = \frac{c' K K_j}{M} \end{aligned}$$

and for $v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c')$,

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{\sigma(t_3)} H_j(t,s) v(s) \Delta s \\ &\geq m_j L b' \int_{t_2}^{\sigma(t_3)} \|G_j(.,s)\| \Delta s = m_j L b' L_j. \end{aligned}$$

and also for $v \in Q(\gamma, \beta, \psi, Ma', a', c')$,

$$\begin{aligned} T_j v(t) &= \int_{t_1}^{\sigma(t_3)} H_j(t,s)v(s)\Delta s \\ &\geq m_j LMa' \int_{s\in I_1} \|G_j(.,s)\|\Delta s = m_j LMa'M_j. \end{aligned}$$

Now, we may apply condition $(A2)$ to get

$$\begin{aligned} \alpha(Tv) &= \min_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\geq M \int_{t_1}^{\sigma(t_3)} G_n(s,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\geq \frac{b'}{K_n} \int_{t_1}^{\sigma(t_3)} G_n(s,s)\Delta s = b'. \end{aligned}$$

Clearly, by condition $(A1)$, we have

$$\begin{aligned} \beta(Tv) &= \max_{t\in I_1} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\leq \frac{a'}{K_n} \int_{t_1}^{\sigma(t_3)} G_n(s,s)\Delta s = a'. \end{aligned}$$

To see that $(B3)$ is satisfied, let $v \in P(\gamma, \alpha, b', c')$ with $\theta(Tv) > \frac{b'}{M}$. Using Lemma 2.4, we get

$$\begin{aligned} \alpha(Tv) &= \min_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\geq M \int_{t_1}^{\sigma(t_3)} G_n(s,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\geq M \max_{t\in[t_1,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &\geq M \max_{t\in[t_2,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s \\ &= M\theta(Tv) > b'. \end{aligned}$$

Finally, we show that $(B4)$ holds. Let $v \in Q(\gamma, \beta, a', c')$ with $\psi(Tv) < Ma'$. In view of Lemma 2.4, we have

$$\beta(Tv) = \max_{t\in I_1} \int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s), T_{n-2}v(s), ..., T_1v(s), v(s))\Delta s$$

$$\leq \max_{t\in[t_1,\sigma(t_3)]}\int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),...,T_1v(s),v(s))\Delta s$$
$$\leq \int_{t_1}^{\sigma(t_3)} G_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),...,T_1v(s),v(s))\Delta s$$
$$= \frac{1}{M}\int_{t_1}^{\sigma(t_3)} MG_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),...,T_1v(s),v(s))\Delta s$$
$$\leq \frac{1}{M}\min_{t\in[t_2,\sigma(t_3)]}\int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),...,T_1v(s),v(s))\Delta s$$
$$\leq \frac{1}{M}\min_{t\in I_1}\int_{t_1}^{\sigma(t_3)} G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),...,T_1v(s),v(s))\Delta s$$
$$= \frac{1}{M}\psi(Tv) < a'.$$

We have proved that all the conditions of Theorem 3.1 are satisfied and so there exist at least three positive solutions $v_1, v_2, v_3 \in \overline{P(\gamma, c')}$ for the BVP (2.5)-(2.6). Therefore, the BVP (1.1)-(1.2) has at least three positive solutions y_1, y_2, y_3 of the form,

$$y_i(t) = T_{n-1}v_i(t) = \int_{t_1}^{\sigma(t_3)} H_{n-1}(t,s)v_i(s)\Delta s, \quad i = 1, 2, 3.$$

This completes the proof. □

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