Strongly k^*-Continuous and Completely k^*-Continuous Function in Ideal Closure Spaces

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Abstract

In this paper, we deal with the concept strongly k^*-continuous function in ideal closure spaces. In particular, we explain the characterization of strongly k^*-continuous function in Ideal Closure Spaces.
Mathematics Subject Classification: 54A05, 54B05, 54C08, 54C10
Keywords: strongly k^*-continuous function, completely k^*-continuous function, restrictions of strongly k^*-continuous function, composite of two strongly k^*-continuous function in ideal closure spaces.

1 Introduction

In 1960, Levine [?] initiated the concept of strongly continuous function in topological spaces. Later many authors generalized some results on the relation between continuous function and strongly continuous function. Characterizations and composite functions of strongly continuous functions were discussed by Shashi Prabha Arya and Ranjana Gupta[?]. Gowri and Pavithra[?] initiated the concept of k^*-continuous funcitons in ideal closure spaces. In this paper, we defined and discussed the properties of strongly k^*-continuous functions in ideal closure spaces. Then the relative results between k^*-continuous function and strongly k^*-continuous function were derived. In addition the composition of two strongly k^*-continuous function and weakly k^*-continuous function were also characterized.

In the last section the concept of completely k^*-continuous function was also introduced and studied. Finally the relation between k^*-continuous function, strongly k^*-continuous function and completely k^*-continuous function were discussed using some examples.

2 Preliminaries

In this section, we recall the basic definitions of ideal closure spaces.

Definition 2.1 *[?] $(X, \Im)$ be a topological space. An Ideal I on a topological space is a collection of non empty collections of subsets of X which satisfies:*
(1) $\emptyset \in I$
(2) $A \in I, B \subseteq A$ implies $B \in I$,
(3) $A \in I, B \in I$ implies $A \cup B \in I$.

If $(X, \Im)$ is a topological space and I is an Ideal on X. Then $(X, \Im, I)$ is called an Ideal topological space or an Ideal space.

Definition 2.2 *[?] Let $P(X)$ be the power set of X. Then the operator $(.)^* : P(X) \to P(X)$ is called a local function of A with respect to $\Im$ and I, is define as follows:For $A \subseteq X$, $A^*(I, \Im^*) = \{x \in X : U \cap A \notin I$ for every open set U containing $x\}$*

Additionally, $cl^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology $\Im^*$. Here $\Im^*$ is finer than $\Im$.

Definition 2.3 *[?] let X be a non-empty set. I be an Ideal on X.*
Let $A^* : P(X) \to P(X)$ be a function of A with respect to I and $\Im$,.
Let $k^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology.
Then the function $k^* : P(X) \to P(X)$ satisfying,

(1) $k^*(\emptyset) = \emptyset$
(2) $A \subset k^*(A)$
(3) $k^*(A \cup B)$=$k^*(A) \cup k^*(B)$ $\forall A, B \subset X$.
(4) $k^*(A) = k^*(k^*(A))$ $\forall A \subset X$ is called a closure operator on X. The structure (X, I, k^*) is called an Ideal Closure Space.

Example 2.4 $X = \{a, b, c\}$ $\Im = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. $I = \{\emptyset, \{c\}\}$

(1) $A = \{a, c\}$ $A^* = \{a, b\}$ $k^*(A) = A \cup A^* \implies k^*\{a, c\}$ =X.
(2) $A = \{b, c\}$ $A^* = \{b\}$ $k^*(A) = A \cup A^* \implies k^*\{b, c\}$ =$\{b, c\}$.
(3) $A = \{a, b\}$ $A^* = \{a, b\}$ $k^*(A) = A \cup A^* \implies k^*\{a, b\}$ =$\{a, b\}$.
(4) $A = X$ $A^* = \{a, b\}$ $k^*(A) = A \cup A^* \implies k^*(X) = X$
(5) $A = \emptyset$ $A^* = \emptyset$ $k^*(A) = A \cup A^* \implies k^*(\emptyset)$ =$\emptyset$.
(6) $A = \{a\}$ $A^* = \{a, b\}$ $k^*(A) = A \cup A^* \implies k^*\{a\}$ =$\{a, b\}$.
(7) $A = \{b\}$ $A^* = \{b\}$ $k^*(A) = A \cup A^* \implies k^*\{b\}$ =$\{b\}$.
(8) $A = \{c\}$ $A^* = \emptyset$ $k^*(A) = A \cup A^* \implies k^*\{c\}$ =$\{c\}$.

Then (X, I, k^*) is an Ideal Closure Space.

Definition 2.5 *[?] A subset A of an Ideal closure space (X, I, k^*) is said to be closed if $k^*(A) = A$.*

Definition 2.6 *[?] A subset A of an Ideal closure space (X, I, k^*) is said to be open if $k^*(X - A) = X - A$ (i.e) $Int^*(A) = A$.*

Definition 2.7 *[?] The set Int A with respect to the closure operator k^* is defined as $Int^*(A) = X - k^*(X - A)(i.e)[k^*(A^C)]^C$, where $A^C = X - A$.*

Definition 2.8 *[?] (X, I, k^*) is an Ideal closure space than the associate topology on X is $\Im^*$ =$\{A^C; k^*(A) = A\}$. Here $\Im$ is not equal to $\Im^*$*

Definition 2.9 *[?] A subset A in an Ideal closure space (X, I, k^*) is called neighbourhood of x if $x \in Int^*(A)$.*

Definition 2.10 *[?] Let (X, I, k^*) be an Ideal closure space. An Closure space (Y, I, k^*_Y) is called a subspace of (X, I, k^*) if $Y \subseteq X$ and $k^*_Y(A) = k^*(A) \cap Y$, $\forall A \subseteq Y$.*

Definition 2.11 *A subset A of an Ideal closure space (X, I, k^*) is said to be regular open if $Int^*(k^*(A)) = A$.*

Definition 2.12 *A subset A of an Ideal closure space (X, I, k^*) is said to be regular closed if $k^*(Int^*(A)) = A$.*

Definition 2.13 *[?] Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. A function f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is said to be k^*-continuous if $f\,(k_1^*(A)) \subseteq k_2^*\,(f(A))$, for every $A \subset X$.*

3 Strongly k^*-Continuous Function In Ideal Closure Spaces

Definition 3.1 *Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. A function $f : (X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is said to be strongly k^*-continuous if for every $A \subset X$, $f(k_1^*(A) \subseteq f(A)$.*

Example 3.2 .
$X = \{a, b, c\}$; $Y = \{x, y, z\}$
$\Im_1 = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$,
$I_1 = \{\emptyset, \{a, b\}\}$; $\Im_2 = \{Y, \{x\}, \{z\}, \{x, z\}\}$; $I_2 = \{\emptyset\}$
$k_1^*(a) = \{a\}$; $k_2^*(x) = \{x, y\}$
$k_1^*(b) = \{b\}$; $k_2^*(y) = \{y\}$
$k_1^*(c) = \{c\}$; $k_2^*(z) = \{y, z\}$
$k_1^*\{a, b\} = \{a, b\}$; $k_2^*\{x, y\} = \{x, y\}$
$k_1^*\{b, c\} = \{b, c\}$; $k_2^*\{y, z\} = \{y, z\}$
$k_1^*\{c, a\} = \{c, a\}$; $k_2^*\{z, x\} = Y$
$k_1^*(X) = X$; $k_2^*(Y) = Y$
$k_1^*(\emptyset) = \emptyset$; $k_2^*(\emptyset) = \emptyset$
f is a mapping from $(X, I_1, k_1^*) \to (X, I_2, k_2^*)$ defined by $f(a) = y, f(b) = z, f(c) = x$. Here f is strongly k^*-continuous function.

Theorem 3.3 *Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. A function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ is strongly k^*-continuous if and only if $f^{-1}(B)$ is closed for all $B \subseteq Y$*

Proof: Let $B \subseteq Y$, f is strongly k^*-continuous function.
To prove that $f^{-1}(B)$ is closed $\implies k_1^*(f^{-1}(B))$=$f^{-1}(B)$. In order to prove above result we have to prove following two conditions,
(1) $f^{-1}(B) \subseteq k_1^*(f^{-1}(B))$
Let $p \in k_1^*(f^{-1}(B))$ implies $p \in f^{-1}(B)$ (by using the properties of ideal closure space $A \subseteq k^*(A)$). Therefore $f^{-1}(B) \subseteq k_1^*(f^{-1}(B))$. Hence (1) is proved.
(2) $k_1^*(f^{-1}(B) \subseteq f^{-1}(B)$
Let $p \in f^{-1}(B)$, $f(P) \in B$, by the definition of strongly k^*-continuous function we get $f(k_1^*(P)) \in B$ then $k_1^*(P) \in f^{-1}(B)$ taking ideal closure operator on both sides we get $k_1^*(k_1^*(P)) \in k_1^*(f^{-1}(B))$ then $k_1^*(P) \in k_1^*(f^{-1}(B))$, $p \in k_1^*(f^{-1}(B))$ therefore $k_1^*(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence (2) is proved.
From (1) and (2) $k_1^*(f^{-1}(B))$=$f^{-1}(B)$. Therefore $f^{-1}(B)$ is closed, for all $B \subseteq Y$.
Conversely, Let $A \subset X$ then $A \subset f^{-1}(f(A))$, $k_1^*(A) \subset k_1^*(f^{-1}f(A))$. Since $f^{-1}(f(A))$

is closed. Therefore $k_1^*(A) \subset f^{-1}(f(A))$, $f(k_1^*(A)) \subset f(A))$. Hence f is strongly k^*-continuous function. □

Theorem 3.4 *A function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ is strongly k^*-continuous if and only if $f^{-1}(B)$ is open for all $B \subseteq Y$.*

Proof: A function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ is strongly k^*-continuous and to prove that $f^{-1}(B)$ is open in order to show that we just prove that $k_1^*(X - f^{-1}(B)) = X - f^{-1}(B)$, let $x \in k_1^*(X - f^{-1}(B))$, $f(x) \in f(k_1^*(X - f^{-1}(B)))$ then $f(x) \in f(X - f^{-1}(B))$ (since f is strongly k^*-continuous), $f(x) \in [f(X) - ff^{-1}(B)]$, $f(x) \in Y - B$ that is $x \in f^{-1}(Y - B)$, $x \in X - f^{-1}(B)$ therefore $k_1^*(X - f^{-1}(B) \subseteq X - f^{-1}(B)$.......(1)
Now let $x \in X - f^{-1}(B)$, $x \in f^{-1}(Y) - f^{-1}(B)$, $x \in f^{-1}(Y - B)$ which gives $f(x) \in Y - B$ by the definition of strongly k^*-continuous function we get $f(k_1^*(x)) \in Y - B$, $k_1^*(x) \in f^{-1}(Y - B)$, $k_1^*(x) \in X - f^{-1}(B)$ then taking ideal closure operator on both sides we get $k_1^*(k_1^*(x)) \in k_1^*(X - f^{-1}(B))$, $k_1^*(x) \in k_1^*(X - f^{-1}(B)$, $x \in k_1^*(X - f^{-1}(B))$........(2)
From (1) and (2) $k_1^*(X - f^{-1}(B)) = X - f^{-1}(B)$. Hence $f^{-1}(B)$ is open in X.
Conversely let $A \subset X$ then $A \subset f^{-1}(f(A))$ then $k_1^*(X - A) \subset k_1^*(X - f^{-1}f(A)) \subset (X - f^{-1}f(A))$, since $f^{-1}(f(A))$ is open, $f(k_1^*(X - A)) \subset f(X - A)$. Hence f is strongly k^*-continuous function. □

Corollary 3.5 *A function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ is strongly k^*-continuous if and only if $f^{-1}(B)$ is both open and closed for all $B \subseteq Y$.*

Proof: The proof follows immediately from theorem 3.3 and theorem 3.4. □

Theorem 3.6 *Let a function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ then the following are equal.*
(i) f is strongly k^*-continuous.
(ii)For every subset $B \subseteq Y$ then $k_1^*(f^{-1}(B)) \subseteq f^{-1}(B)$.

Proof: .
$(i) \implies (ii)$
Let $B \subseteq Y$ then $f^{-1}(B) \subseteq X$. Since f is strongly k^*-continuous we get $f(k_1^*(f^{-1}(B))) \subseteq f(f^{-1}(B)) \subseteq B$ therefore $f^{-1}(f(k_1^*(f^{-1}(B)))) \subseteq f^{-1}(B)$. Hence $k_1^*(f^{-1}(B)) \subseteq f^{-1}(B)$.
$(ii) \implies (i)$
$A \subset X$, $B \subset Y$, let a function $f : (X, I_1, k_1^*) \to (X, I_2, k_2^*)$ then $f(A) \subseteq B$, $A \subseteq f^{-1}(B)$ from (ii) $B \subseteq Y$, $k_1^*(f^{-1}(B)) \subseteq f^{-1}(B)$, $f(k_1^*(f^{-1}(B))) \subseteq B$, $f(k_1^*(A)) \subseteq f(A)$. Therefore f is strongly k^*-continuous. □

Theorem 3.7 *Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. Let (A, I, k_A^*) be a subspace of (X, I_1, k_1^*). if f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is strongly k^*-continuous then $f_{|A}$: $(A, I, k_A^*) \to (Y, I_2, k_2^*)$ is strongly k^*-continuous.*

Proof: Let $A \subseteq X$, $B \subset Y$. Let $(f_{|A})^{-1}(B) = f^{-1}(B) \cap A$. Since f is strongly k^*-continuous therefore $f^{-1}(B)$ is both open and closed in X. It follows that $f^{-1}(B) \cap A$ is also relatively both open and closed in A. Hence $f_{|A}$ is strongly k^*-continuous. □

Theorem 3.8 *Let (X, I_1, k_1^*), (Y, I_2, k_2^*) and (Z, I_3, k_3^*) be ideal closure spaces. Let f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is strongly k^*-continuous and g: $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ be any function then $g \circ f$:$(X, I_1, k_1^*) \to (Z, I_3, k_3^*)$ is strongly k^*-continuous.*

Proof: Let A be any subset of Z. Then $g^{-1}(A)$ is a subset of Y. Since f is strongly k^*-continuous therefore $f^{-1}(g^{-1}(A))$ is both open and closed subset of X that is $g^{-1}(f^{-1}(A))$ is both open and closed then $(g \circ f)^{-1}(A)$ is both open and closed subset of X. Hence $g \circ f$ is strongly k^*-continuous. □

Definition 3.9 *A function f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is said to be weakly k^*-continuous if for each point $x \in X$ and each open set H containing $f(x)$ there is an open set G containing x such that $f(G) \subset k_2^*(H)$.*

Theorem 3.10 *If f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is a weakly k^*-continuous function and g: $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ is strongly k^*-continuous then $g \circ f$:$(X, I_1, k_1^*) \to (Z, I_3, k_3^*)$ is strongly k^*-continuous.*

Proof: Let A be any subset of Z. Since g is strongly k^*-continuous therefore $g^{-1}(A)$ is both open and closed subset of Y. Since f is weakly k^*-continuous and $g^{-1}(A)$ is an open subset of Y therefore $k_1^*[f^{-1}(g^{-1}(A))] \subseteq f^{-1}[k_1^*(g^{-1}(A))] = f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$. It follows that $(g \circ f)^{-1}(A)$ is a closed set of X. Hence $g \circ f$ is strongly k^*-continuous. □

Theorem 3.11 *If f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is strongly k^*-continuous function and g: $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ is k^*-continuous then $g \circ f$:$(X, I_1, k_1^*) \to (Z, I_3, k_3^*)$ is k^*-continuous.*

Proof: Let f is strongly k^*-continuous function then $f(k_1^*(A)) \subset f(A)$, $A \subset X$
Let g is k^*-continuous function then $g(k_2^*(A)) \subset k_3^*(g(A))$, $A \subset X$. Now we have to prove that $g \circ f$ is k^*-continuous function that is $g \circ f(k_1^*(A)) \subset k_3^*(g \circ f(A))$. Let $g(f(k_1^*(A))) \subset g(f(A))$, where f is strongly k^*-continuous then $g(f(A)) \subset k_3^*(g(f(A))) \subset k_3^*(g \circ f(A))$, where g is k^*-continuous. Therefore $g \circ f(k_1^*(A)) \subset k_3^*(g \circ f(A))$. Hence $g \circ f$ is k^*-continuous function. □

Theorem 3.12 *If f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is k^*-continuous function and g: $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ is strongly k^*-continuous then $g \circ f$:$(X, I_1, k_1^*) \to (Z, I_3, k_3^*)$ is strongly k^*-continuous.*

Proof: Let f is k^*-continuous then $f(k_1^*(A)) \subset k_2^*(f(A))$, $A \subset X$.
Let g is strongly k^*-continuous then $g(k_2^*(A)) \subset g(A)$, $A \subset X$. To prove that $g \circ f(k_1^*(A)) \subset g \circ f(A)$, $A \subset X$. Now let $g(f(k_1^*(A))) \subset g(k_2^* f(A)) \subset g(f(A)) \subset g \circ f(A)$, $A \subset X$. Therefore $g \circ f(k_1^*(A)) \subset g \circ f(A)$. Hence $g \circ f$ is strongly k^*-continuous function. □

Theorem 3.13 *If a function f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is strongly k^*-continuous then it is k^*-continuous.*

Proof: Let f is strongly k^*-continuous. For every subset B in Y then $f^{-1}(B)$ is both open and closed. Therefore inverse image of every open set in Y is open in X. Hence f is k^*-continuous. □

Remark 3.14 *But converse of the above theorem is not true shown in the following example*

Example 3.15 $X = \{a, b, c\}$; $Y = \{x, y, z\}$
$\Im_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$; $\Im_2 = \{Y, \{x\}, \{z\}, \{x, z\}\}$;
$I_1 = \{\emptyset, \{c\}\}$; $I_2 = \{\emptyset, \{y\}\}$
$k_1^*(a) = \{a, b\}$; $k_2^*(x) = \{x, y\}$
$k_1^*(b) = \{b\}$; $k_2^*(y) = \{y\}$
$k_1^*(c) = \{c\}$; $k_2^*(z) = \{y, z\}$
$k_1^* \{a, b\} = \{a, b\}$; $k_2^* \{x, y\} = \{x, y\}$
$k_1^* \{b, c\} = \{b, c\}$; $k_2^* \{y, z\} = \{y, z\}$
$k_1^* \{c, a\} = X$; $k_2^* \{z, x\} = Y$
$k_1^*(X) = X$; $k_2^*(Y) = Y$
$k_1^*(\emptyset) = \emptyset$; $k_2^*(\emptyset) = \emptyset$
f is a mapping from $(X, I_1, k_1^*) \to (X, I_2, k_2^*)$ defined by $f(a) = x, f(b) = y, f(c) = Z$. Here f is k^*-continuous function then let $A = a$ implies $f(k_1^*(a)) \not\subset f(a)$ then $f(a, b) \not\subset \{x\}$ that is $\{x, y\} \not\subset \{x\}$. Therefore f is not strongly k^*-continuous.

Proposition 3.16 *Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. A map f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ be a bijection then the following statements are equal.*
(i) The inverse map $f^{-1} : (Y, I_2, k_2^*) \to (X, I_1, k_1^*)$ is strongly k^*-continuous.
(ii) f is both open map and closed map.

Proof: .
$(i) \implies (ii)$
Let $f^{-1} : (Y, I_2, k_2^*) \to (X, I_1, k_1^*)$ is strongly k^*-continuous and let B be any subset in X then $(f^{-1})^{-1}(B)$ is both open and closed in Y which implies $f(B)$ is both open and closed. Thus (1) $\implies (ii)$.
$(ii) \implies (i)$
Let f is both open and closed then $BinX$, $f(B)$ is both open and closed but $f(B)$=$(f^{-1})^{-1}(B)$. Thus f^{-1} is strongly k^*-continuous. □

4 Completely k^*-Continuous Functions in Ideal Closure Spaces

Definition 4.1 *Let (X, I_1, k_1^*) and (Y, I_2, k_2^*) be ideal closure spaces. A function f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is said to be completely k^*-continuous at a point $x \in X$ if for every neighbourhood M of $f(x)$ there is a regular open neighbourhood N of x such that $f(N) \subset M$.*

Theorem 4.2 *Let a map f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is completely k^*-continuous if and only if inverse image of every open subset of Y is a regular open subset of X.*

Proof: Assume that f is completely k^*-continuous and let B be any open set in Y we have to show that $f^{-1}(B)$ is regular open in X. If $f^{-1}(B) = \emptyset$ there is nothing to prove so let $f^{-1}(B) \neq \emptyset$ and let $x \in f^{-1}(B)$ so that $f(x) \in B$ by completely k^*-continuous of f, there exist an regular open set N_x in X such that $x \in N_x$ and

$f(N_x) \subset B$ that is $x \in N_x \subset f^{-1}(B)$. This shows that $f^{-1}(B)$ is a regular open neighbourhood in X.
Conversely let $f^{-1}(B)$ is regular open in X for every open set B in Y. We shall show that f is completely k^*-continuous at $x \in X$. Let B be any open set in Y such that $f(x) \in B$ so that $x \in f^{-1}(B)$, by hypothesis $f^{-1}(B)$ is regular open in X. If $f^{-1}(B) = N$ then N is regular open set in X containing x such that $f(N) = f(f^{-1}(B)) \subset B$ implies $f(N) \subset B$. Hence f is completely k^*-continuous. □

Theorem 4.3 *Let a map f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is completely k^*-continuous if and only if inverse image of every closed subset of Y is a regular closed subset of X.*

Proof: .
Assume that f is completely k^*-continuous and let F be any closed set in Y. To show that $f^{-1}(F)$ is regular closed in X and $Y - F$ is open in Y, $f^{-1}(Y - F) = X - f^{-1}(F)$, since f is completely k^*-continuous therefore $f^{-1}(Y - F)$ is regular open in X then $X - f^{-1}(F)$ is regular open that is $f^{-1}(F)$ is regular closed in X.
Conversely let $f^{-1}(F)$ is regular closed in X for every closed set F in Y. We want to show that f is completely k^*-continuous. Let B be any open set in Y. Then $Y - B$ is closed in Y and so by hypothesis $f^{-1}(Y - B) = X - f^{-1}(B)$ is regular closed in X that is $f^{-1}(B)$ is regular open in X. Hence f is completely k^*-continuous by using previous theorem. □

Lemma 4.4 *In a ideal closure spaces (X, I, k^*), U is regular open implies U is open.*

Proof: If U is regular open
$\implies U = int^*(k^*(U))$
$\implies U = X - k^*(X - U)$
$\implies X - U = k^*(X - U)$
$\implies$ U is open.
There fore if f is regular open then U is open. □

Theorem 4.5 *If f is completely k^*-continuous then each point $x \in X$ and for each open neighbourhood of $f(x)$ there is a open neighbourhood N of x such that $f(N) \subset B$.*

Proof: If f is completely k^*-continuous. Since B is open therefore $Y \setminus B$ is closed and consequently $f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B)$ is regular closed that is $f^{-1}(B)$ is regular open. Also $x \in f^{-1}(B) = N$(say). Here N is regular open neighbourhood of x which implies N is open neighbourhood of x by using previous lemma then $f(N) \subset M$.
Obviously, every strongly k^*-continuous function is completely k^*-continuous and every completely k^*-continuous mapping is k^*-continuous. The converse implications do not held as is shown by the following example. □

Example 4.6 *Let $X = \{a, b, c, d\}$, $\Im_1 = \{X, \emptyset, \{a, b\}, \{c, d\}\}$, $I_1 = \{\emptyset, \{c\}, \{a, c\}\}$. Then ideal closure space (X, I_1, k_1^*) is defined by $k_1^*(a) = \{a, b\}$, $k_1^*(b) = \{a, b\}$, $k_1^*(c) = \{c\}$, $k_1^*(d) = \{c, d\}$, $k_1^*(a, b) = \{a, b\}$, $k_1^*(b, c) = \{a, b, c\}$, $k_1^*(c, d) = \{c, d\}$, $k_1^*(a, d) = X$, $k_1^*(c, a) = \{a, b, c\}$, $k_1^*(b, d) = X$ $k_1^*(a, b, c) = \{a, b, c\}$, $k_1^*(b, c, d) =$*

$k^*(a,c,d) = k_1^*(a,b,d) = X$.
Let $Y = \{x,y,z\}$, $\Im_2 = \{Y, \emptyset, \{a,c\}\}$, $I_2 = \{\emptyset, \{b\}\}$.
(X, I_2, k_2^*) *is followed by,*
$k_2^*(x) = Y$; $k_2^*(y) = \{y\}$; $k_2^*(z) = Y$; $k_2^*\{x,y\} = Y$; $k_2^*\{y,z\} = Y$; $k_2^*\{z,x\} = Y$; $k_2^*(Y) = Y$; $k_2^*(\emptyset) = \emptyset$.
f *is a mapping from* $(X, I_1, k_1^*) \to (X, I_2, k_2^*)$ *defined by* $f(a) = x, f(b) = z, f(c) = z, f(d) = x$. *Here* f *is completely* k^**-continuous function. Now let* $A = \{a\}$ *using the definition of strongly* k^**-continuous function* $f(k_1^*(a)) \not\subset f(a)$. *Therefore* f *is not strongly* k^**-continuous function.*

Example 4.7 .
Let $X = \{a,b,c,d\}$, $\Im_1 = \{X, \emptyset, \{a,c,d\}, \{c\}, \{a\}\}$, $I_1 = \{\emptyset, \{b\}\}$. *Then ideal closure space* (X, I_1, k_1^*) *is defined by* $k_1^*(a) = \{a,b,d\}$, $k_1^*(b) = \{b\}$, $k_1^*(c) = \{b,c,d\}$, $k_1^*(d) = \{b,d\}$, $k_1^*(a,b) = \{a,b,d\}$, $k_1^*(b,c) = \{b,c,d\}$, $k_1^*(c,d) = \{b,c,d\}$, $k_1^*(a,d) = \{a,d\}$, $k_1^*(c,a) = X$,$k_1^*(b,d) = X$, $k_1^*(a,b,c) = X$, $k_1^*(b,c,d) = \{b,c,d\}$ $k^*(a,c,d) = X$ $k_1^*(a,b,d) = \{a,b,d\}$, $k_1^*(X) = X$, $k_1^*(\emptyset) = \emptyset$
Let $Y = \{x,y,z,t\}$, $\Im_2 = \{X, \emptyset, \{x,y\}, \{x\}, \{y\}\}$, $I_2 = \{\emptyset, \{z\}\}$. *Then ideal closure space* (Y, I_2, k_2^*) *is defined by* $k_2^*(x) = \{x,y,t\}$, $k_2^*(y) = \{y,z,t\}$, $k_2^*(z) = \{z\}$, $k_2^*(d) = \{z,t\}$, $k_2^*(x,y) = Y$, $k_2^*(y,z) = Y$, $k_2^*(z,t) = \{z,t\}$, $k_2^*(x,t) = \{x,z,t\}$, $k_2^*(z,x) = \{x,z,t\}$, $k_2^*(y,t) = \{y,z,t\}$ $k_2^*(x,y,z) = X$, $k_2^*(y,z,t) = \{y,z,t\}$ $k_2^*(x,z,t) = \{z,z,t\}$ $k_1^*(x,y,t) = X$, $k_2^*(Y) = Y$, $k_2^*(\emptyset) = \emptyset$
f *is a mapping from* $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ *defined by* $f(a) = x, f(b) = z, f(c) = y, f(d) = t$. *Here* f *is* k^**-continuous function but not completely* k^**-continuous function.*

Example 4.8 *Consider the function* f *in Example 4.6. Here* f *is completely* k^**-continuous function. Let the restriction of* X *is defined by* $X\big|_A = \{b,c,d\}$ *and ideal closure space* $(X\big|_A, I_1, k_{X|A}^*)$ *is defined by*
$k_{X|A}^*(b) = \{b\}$; $k_{X|A}^*(c) = \{c\}$; $k_{X|A}^*(d) = \{d\}$; $k_{X|A}^*\{b,c\} = \{b,c\}$; $k_{X|A}^*\{c,d\} = X\big|_A$; $k_{X|A}^*\{d,b\} = X\big|_A$; $k_{X|A}^*(X\big|_A) = X\big|_A$; $k_{X|A}^*(\emptyset) = \emptyset$.
Now the restriction of f is defined by as follows
$f\big|_A : (X\big|_A, I_1, k_{X|A}^*) \to (X, I_2, k_2^*)$ *then* $f(b) = z, f(c) = z, f(d) = x$. *Here* $f\big|_A$ *is not completely* k^**-continuous function. Hence the restriction of a completely* k^**-continuous function may fail to be completely* k^**-continuous.*

Theorem 4.9 *If f:* $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ *is completely* k^**-continuous and g:* $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ *is* k^**-continuous then* $g \circ f : (X, I_1, k_1^*) \to (Z, I_3, k_3^*)$ *is completely* k^**-continuous.*

Proof: Let U be any open subset of Z. Since g is k^*-continuous therefore $g^{-1}(U)$ is an open subset of Y. Since f is completely k^*-continuous therefore $f^{-1}(g^{-1}(U))$ is a regular open subset of X that is $(g \circ f)^{-1}(U)$ is regular open subset of X. Hence $g \circ f$ is completely k^*-continuous. □

Corollary 4.10 *The composite of two completely* k^**-continuous mapping is completely* k^**-continuous.*

Definition 4.11 *A mapping f:* $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ *is said to be almost open if the image of every regular open set is open.*

Theorem 4.12 *If f: $(X, I_1, k_1^*) \to (Y, I_2, k_2^*)$ is almost open and onto with completely k^*-continuous and g: $(Y, I_2, k_2^*) \to (Z, I_3, k_3^*)$ is a mapping such that $g \circ f$ is completely k^*-continuous then g is k^*-continuous.*

Proof: Let G be any open subset of Z. Since $g \circ f$ is completely k^*-continuous therefore $(g \circ f)^{-1}(G)$ is regular open in X. Since f is almost open therefore $f((g \circ f)^{-1}(G))$ is an open subset of Y that is $f(f^{-1}(g^{-1}(G))) = g^{-1}(G)$ is an open subset of Y. Hence g is k^*-continuous. □

5 Conclusion

In this paper, the important concepts of strongly k^*-continuous function and completely k^*-continuous functions in ideal closure space were introduced. Also the restriction of strongly k^*-continuous function and composite of two strongly k^*-continuous functions are analyzed.

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