Modified class of Ratio-Cum-Dual to Ratio Estimators for Estimating Finite Population Mean Using Auxiliary Information

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Abstract

In this manuscript, a class of dual to ratio and product estimators is suggested, and its Mean-Squared Error (MSE) is derived. The results are proved theoretically and empirically.

Keywords: exponential estimator, dual to ratio estimator, auxiliary variable, mean square error (MSE).

1. Introduction

Cochran (1940) laid emphasis on previous information at estimation stage for the estimating population parameters. Robson (1957) gave product estimator which was revised by Murthy (1964). Many authors such as Sisodia and Dwivedi (1981), Upadhya and Singh (1999) and Singh and Tailor (2003), used known value of auxiliary variate for estimation techniques.

2. Some existing estimators

The ratio estimator for $\bar{Y}$ is given as,

$$\bar{y}_R = \frac{\bar{y}}{\bar{x}}\bar{X}, \tag{1}$$

$\bar{y}$ is the sample mean of study variable, $\bar{x}$ is the sample mean of the auxiliary variable. C_y, C_x are the coefficient of variation of y and x respectively and ρ the correlation coefficient.

The MSE of above estimator is

$$MSE(\bar{y}_R) = \gamma \bar{Y}^2\left[C_y^{\,2} + C_x^{\,2} - 2\rho C_x C_y\right]. \tag{2}$$

The classical product estimator for $\bar{Y}$ is given by,

$$\bar{y}_P = \bar{y}\frac{\bar{x}}{\bar{X}}, \tag{3}$$

and the MSE is given by

$$MSE(\bar{y}_P) = \gamma \bar{Y}^2 \left[C_y^{\ 2} + C_x^{\ 2} + 2\rho C_x C_y \right]. \tag{4}$$

Srivenkataramana (1980) and Bandyopadhyaya (1980) attained dual to ratio and product estimators respectively as

$$\bar{y}_R^{\ *} = \bar{y} \left(\frac{\bar{x}^*}{\bar{X}} \right), \tag{5}$$

$$\bar{y}_P^{\ *} = \bar{y} \left(\frac{\bar{X}}{\bar{x}^*} \right). \tag{6}$$

The bias and *MSE* of $\bar{y}_R^{\ *}$ and $\bar{y}_P^{\ *}$ are respectively given as,

$$B(\bar{y}_R^{\ *}) = -\frac{(1-f)}{n} \bar{Y} C_x^{\ 2} g K \tag{7}$$

$$B(\bar{y}_P^{\ *}) = -\frac{(1-f)}{n} \bar{Y} C_x^{\ 2} g (g + K) \tag{8}$$

$$MSE(\bar{y}_R^{\ *}) = -\frac{(1-f)}{n} \bar{Y}^2 \left[C_y^{\ 2} + C_x^{\ 2} g (g - 2K) \right] \tag{9}$$

$$MSE(\bar{y}_P^{\ *}) = -\frac{(1-f)}{n} \bar{Y}^2 \left[C_y^{\ 2} + C_x^{\ 2} g (g + 2K) \right] \tag{10}$$

Exponential ratio estimator of $\bar{Y}$ proposed by Bahl and Tuteja (1991) is

$$\bar{y}_{ER} = \bar{y} \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right), \tag{11}$$

and the MSE is given by

$$MSE(\bar{y}_{ER}) = \gamma \bar{Y}^2 \left[C_y^{\ 2} + \frac{C_x^{\ 2}}{4} - 2\rho C_x C_y \right], \tag{12}$$

and the exponential product estimator is given by

$$\bar{y}_{EP} = \bar{y} \exp\left(\frac{\bar{x} - \bar{X}}{x + \bar{X}} \right), \tag{13}$$

and the MSE is given by

$$MSE(\bar{y}_{EP}) = \gamma \bar{Y}^2 \left[C_y{}^2 + \frac{C_x{}^2}{4} + 2\rho C_x C_y \right]. \quad (14)$$

Tailor and Tailor in (2012) gave exponential dual of ratio and dual of product estimator given as,

$$\bar{y}_{ER}{}^* = \bar{y} \exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}} \right), \quad (15)$$

$$\bar{y}_{EP}{}^* = \bar{y} \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*} \right) \quad (16)$$

the MSE is given by,

$$MSE(\bar{y}_{ER}{}^*) = \gamma.\bar{Y}^2 \left(C_y^2 + gC_x^2 \left(\frac{g}{4} - K_{yx} \right) \right). \quad (17)$$

$$MSE(\bar{y}_{EP}{}^*) = \gamma.\bar{Y}^2 \left(C_y^2 + gC_x^2 \left(\frac{g}{4} + K_{yx} \right) \right) \quad (18)$$

3. Proposed estimator

We have suggested generalized class of dual of the ratio cum product estimator of $\bar{Y}$ given as

$$t^* = \bar{y} \left(\frac{\bar{x}^*}{\bar{X}} \right)^{\alpha} \exp\left\{ \frac{(a\bar{X} + b) - (a\bar{x}^* + b)}{(a\bar{X} + b) + (a\bar{x}^* + b)} \right\} \quad (19)$$

where $x_i{}^* = \dfrac{N\bar{X} - nx_i}{N - n}$

The associated sample mean is obtained as

$$\bar{x}^* = (1 + g)\bar{X} - g\bar{x} \text{ and } g = \frac{n}{N - n}$$

In order to get the (MSE) of the proposed estimator, we write

$$\bar{y} = \bar{Y}(1 + e_0) \qquad \bar{x} = \bar{X}(1 + e_1)$$

Such that

$$E(e_0) = E(e_1) = 0$$

and

$$E\left(e_0{}^2\right)=Var\left(\bar{y}\right)=\gamma.C_y{}^2, \quad E\left(e_1{}^2\right)=Var\left(\bar{x}\right)=\gamma.C_x{}^2, \quad E\left(e_0e_1\right)=Cov\left(\bar{y},\bar{x}\right)=\gamma\rho C_yC_x.$$

Expressing (15) in e's we get

$$t^* = \bar{Y}\left(1+e_0\right)\left(1-ge_1\right)^{\alpha}\exp\left\{\frac{ag\bar{X}e_1}{2\left(a\bar{X}+b\right)-g\bar{X}e_1}\right\} = \bar{Y}\left(1+e_0\right)\left(1-ge_1\right)^{\alpha}\exp\left\{\frac{g\theta e_1}{1-g\theta e_1}\right\}$$

$$= \bar{Y}\left(1+e_0\right)\left(1-ge_1\right)^{\alpha}\exp\left\{g\theta e_1\left(1-g\theta e_1\right)^{-1}\right\} \tag{20}$$

Where $\theta=\dfrac{a\bar{X}}{2\left(a\bar{X}+b\right)}$

Expanding the right-hand side of (16) and ignoring the terms of e's with higher power we get

$$t^* = \bar{Y}\left(1+e_0\right)\left[1+g\theta e_1+g^2\theta^2e_1{}^2+\alpha g^2\theta e_1{}^2-\alpha ge_1+\frac{\alpha\left(\alpha-1\right)}{2}g^2e_1{}^2\right] \tag{21}$$

Subtracting $\bar{Y}$ and taking expectations of (21) we get,

$$E\left(t^*-\bar{Y}\right)=E\left[\bar{Y}\left\{e_0+g\theta e_1-\alpha ge_1\right\}\right] \tag{22}$$

Squaring the equation (22) and solving, gives MSE,

$$E\left(t^*-\bar{Y}\right)^2=E\left[\bar{Y}^2\left\{e_0+g\theta e_1-\alpha ge_1\right\}^2\right]$$

$$MSE\left(t^*\right)=\bar{Y}^2E\left(e_0{}^2+g^2\theta^2e_1{}^2+g^2\alpha^2e_1{}^2+2\theta ge_0e_1-2\alpha ge_0e_1-2\alpha\theta g^2e_1{}^2\right)$$

$$MSE\left(t^*\right)=\bar{Y}^2\gamma\left[C_y{}^2+g^2\theta^2C_x{}^2+g^2\alpha^2C_x{}^2+2g\theta\rho C_xC_y-2g\alpha\rho C_xC_y-2g^2\alpha\theta C_x{}^2\right] \tag{23}$$

We differentiate the $MSE\left(t^*\right)$ with respect to α, the α_{opt} is given as,

$$\alpha_{opt}=\left(\frac{\rho C_y}{gC_x}+\theta\right) \tag{24}$$

Using the value of α_{opt}, we get the $MSE_{\min}\left(t^*\right)$ given as

$$MSE_{\min}\left(t^*\right)=\gamma\bar{Y}^2C_y{}^2\left(1-\rho^2\right) \tag{25}$$

The suggested estimator t^* at its optimum condition is efficient. Various estimators can be obtained by allocating suitable values of unknown parameters.

Table 1

Some Members of Suggested estimator t^*		
Values of		Estimators
a	b	
1	1	$t_1^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)$
1	$\beta_2(x)$	$t_2^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*+2\beta_2(x)}\right)$
1	C_x	$t_3^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*+2C_x}\right)$
1	ρ_{yx}	$t_4^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*+2\rho_{yx}}\right)$
$\beta_2(x)$	C_x	$t_5^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\beta_2(x)(\bar{X}-\bar{x}^*)}{\beta_2(x)(\bar{X}+\bar{x}^*)+2C_x}\right)$
C_x	$\beta_2(x)$	$t_6^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{C_x(\bar{X}-\bar{x}^*)}{C_x(\bar{X}+\bar{x}^*)+2\beta_2(x)}\right)$
C_x	ρ_{yx}	$t_7^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{C_x(\bar{X}-\bar{x}^*)}{C_x(\bar{X}+\bar{x}^*)+2\rho_{yx}}\right)$
ρ_{yx}	C_x	$t_8^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\rho_{yx}(\bar{X}-\bar{x}^*)}{\rho_{yx}(\bar{X}+\bar{x}^*)+2C_x}\right)$
$\beta_2(x)$	ρ_{yx}	$t_9^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\beta_2(x)(\bar{X}-\bar{x}^*)}{\beta_2(x)(\bar{X}+\bar{x}^*)+2\rho_{yx}}\right)$
ρ_{yx}	$\beta_2(x)$	$t_{10}^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\rho_{yx}(\bar{X}-\bar{x}^*)}{\rho_{yx}(\bar{X}+\bar{x}^*)+2\beta_2(x)}\right)$

4. Efficiency comparison

In this section, we have compared the *MSE* of $\bar{y}$, $\bar{y}_R^*$, $\bar{y}_P^*$ $\bar{y}_{ER}^*$ and $\bar{y}_{EP}^*$ with t^* given as,

$$\left[Var(\bar{y}) - MSE_{\min}\left(t^*\right)\right] \geq 0$$

$$\left[MSE\left(\bar{y}_R^*\right) - MSE_{\min}\left(t^*\right)\right] \geq 0$$

$$\left[MSE\left(\bar{y}_P^*\right) - MSE_{\min}\left(t^*\right)\right] \geq 0$$

$$\left[MSE\left(\bar{y}_{ER}^*\right) - MSE_{\min}\left(t^*\right)\right] \geq 0$$

$$\left[MSE\left(\bar{y}_{EP}^*\right) - MSE_{\min}\left(t^*\right)\right] \geq 0$$

Theoretically, it is established that our estimator is the best estimators, because above conditions are always satisfied.

5. Numerical illustration

To show the efficiency of our estimator, we consider two data sets given in Table 2. Table 3 shows Percent Relative Efficiency (PRE).

Population 1:

For numerical illustration the data set given by Murthy (1967). Where X is the fixed capital and Y is the output of the 80 factories.

Population 2:

Here we have considered the data given by Kadilar and Cingi (2006). Where y is the level of apple production and x is the number of apple trees. Table 2 shows the data statistics.

Table 2

Parameters	Population 1	Population 2
N	80	104
n	20	20
$\bar{Y}$	51.824	625.37
$\bar{X}$	11.264	13.93

ρ	0.9413	0.865
C_y	0.3542	1.866
C_x	0.7507	1.653

The PRE's of the suggested and that of the existing estimators is given in table 3.

Table 3

Estimators	Population 1	Population 2
$\bar{y}$	100.000	100.000
$\bar{y}_R^*$	588.81	147.08
$\bar{y}_{ER}^*$	216.97	120.64
$t_i^*(proposed)$	**694.71**	**396.83**

6. Conclusion

The PRE's of the estimators in Table 3. shows that our estimator $t_i^*(proposed)$ performed well than the other estimators. Hence, it is preferred to use this estimator in practice.

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