Rectangular Metric-like Space and Common Fixed Points without Continuity of Mappings

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Abstract

In this present paper, we establish a common fixed point theorem for two self-maps in rectangular metric-like space, which generalizes and extends the results of Mlaiki et al. [6]. For the existence of fixed points, it is not necessary that the mappings involved are continuous. An example is provided in support of our main result.

Keywords

Metric-like space, Common fixed point, Rectangular metric-like Space, Cauchy sequence

1 Introduction

Fixed point theory, as the name suggests, is a branch of non-linear analysis deals with finding the solutions of various problems of social and natural sciences, using the concept of fixed points. Or in simple words, we can say that, it is the phenomenon that helps in finding out the solution of non-linear equation which takes the form Tx = x, where T is a self-mapping which is defined on an appropriate subset of a metric space or on some generalized subspaces of metric space.

Fixed point theory, now a days, is the flourishing area of research. Due to its application in various disciplines, a number of authors have given their contribution with several publications and many more are actively working on this research subject.

The most useful result of fixed point theory is contraction mapping principle which is known as the 'Banach contraction principle'. This auspicious principle came into existence in 1922 in Banach's thesis, at the time, when Banach was trying to find out the possibilities of existence of the solution of an integral equation.

Banach's contraction principle states that a contraction mapping of a complete metric space into itself is necessarily continuous and has a unique fixed point.

Now, the next question was that if there exists any contraction map in which continuity is not necessarily required.

Kannan [5] was the first one to work on this problem and found that in the following inequality the continuity of the contraction mapping is not necessary for having unique fixed point. He gave the following contraction:

$$d(Tu, Tv) \leq \alpha \,(\, d(u, Tv) + d(v, Tu) \,) \qquad \text{for all } u, v \in U \text{ and } 0 \leq \alpha \leq \frac{1}{2}$$

where T is a single self-map. After Kannan, many research papers were given on the discontinuity of the mappings under various contraction maps. Many worked on a pair or two pairs of mappings which were discontinuous but were having fixed points. The aim of this paper is to provide a unique common fixed point result in rectangular metric-like space for two self-maps which are not continuous.

2 Preliminaries:

Definition 2.1[4]. Let U be a non-empty set. A function $\sigma: U\times U \rightarrow [0, \infty)$ is said to be a metric-like (dislocated metric) on U if for any u, v, w ∈ U, the following conditions hold:

(σ1) $\sigma(u, v) = 0 \Rightarrow u = v$;

(σ2) $\sigma(u, v) = \sigma(v, u)$;

(σ3) $\sigma(u, v) \leq \sigma(u, w) + \sigma(w, v)$.

Then the pair (U, σ) with σ as metric-like is called as **metric-like (or a dislocated metric) space.**

Definition 2.2[6]. Let U be a non-empty set and $\rho_r : U\times U \rightarrow [0,\infty)$ be a function. If the following conditions are satisfied for all u, v in U and x, y ∈ X \ {u, v}:

(1) $\rho_r(u, v) = 0 \Rightarrow u = v$;
(2) $\rho_r(u, v) = \rho_r(v, u)$;
(3) $\rho_r(u, v) \leq \rho_r(u, x) + \rho_r(x, y) + \rho_r(y, v)$; (*rectangular inequality*)

then the pair (U, ρ_r) is called a **rectangular metric-like space.**

Example 2.3. Let U = {1, 2, 3, 4} and define the function $\rho_r : U\times U\rightarrow[0,\infty)$ by

$$\rho_r(u, v) = \begin{Bmatrix} 2 & \text{for } u \neq v \\ 4 & \text{for } u = v = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

Then, it is clear that the conditions (1) and (2) of Definition (2.2) are satisfied. So after that we need to verify the last condition of definition (2.2).

For all x, y ∈ U \ {u, v}, we have

$\rho_r(u, x) + \rho_r(x, y) + \rho_r(y, v) = 2 + \rho_r(x, y) + 2 = 4 + \rho_r(x, y) \geq \rho_r(u, v)$, for all u, v ∈ U.

Therefore, (U, ρ_r) is a Rectangular metric-like space as all the conditions of definition (2.2) are satisfied.

Definition 2.4[6]. Let (U, ρ_r) be a rectangular metric-like space. Then

(i) A sequence $\{u_n\}$ is called **ρ_r – convergent** if there exists u ∈ U such that $\lim_{n\to\infty} \rho_r(u_n, u) = \rho_r(u, u)$.

(ii) A sequence $\{u_n\}$ is called **ρ_r – Cauchy** if and only if $\lim_{n,m\to\infty} \rho_r(u_n, u_m)$ exists and finite.

(iii) (U, ρ_r) is called **ρ_r – complete** if every ρ_r – Cauchy sequence is ρ_r - convergent.

3 Main Result

Theorem 3.1: Let (U, ρ_r) be a complete rectangular metric-like space and let S, T: U → U be mappings such that

$$\rho_r(Su, Tv) \leq h \max\left\{ \rho_r(u, v), \rho_r(u, Su), \rho_r(v, Tv), \frac{(\rho_r(u, Tv) + \rho_r(v, Su))}{2} \right\} \quad \ldots\ldots (3.1.1)$$

for all u, v ∈ U and 0 < h < 1. Then S and T have unique common fixed point.

Proof: To obtain unique common fixed point, we consider two cases:

Case 1: To obtain common fixed point.

Case 2: The uniqueness of the common fixed point.

Case 1: Let $u_0 \in U$. We have taken this point, as fixed point can always be found by using Picard iteration beginning with some initial choice and we have taken that initial choice as $u_0 \in U$.

Define the sequence $\{u_n\}$ by

$u_1 = Su_0$, $u_2 = Tu_1$, $u_3 = Su_2$, $u_4 = Tu_3$, … , $u_{2n} = Tu_{2n-1}$, $u_{2n+1} = Su_{2n}$, …

Now, consider

$\rho_r(u_{2n+1}, u_{2n+2}) = \rho_r(Su_{2n}, Tu_{2n+1})$

$\leq h$ max

$$\left\{ \rho_r(u_{2n}, u_{2n+1}), \rho_r(u_{2n}, Su_{2n}), \rho_r(u_{2n+1}, Tu_{2n+1}), \frac{(\rho_r(u_{2n}, Tu_{2n+1}) + \rho_r(u_{2n+1}, Su_{2n}))}{2} \right\}$$

$= h$ max

$$\left\{ \rho_r(u_{2n}, u_{2n+1}), \rho_r(u_{2n}, u_{2n+1}), \rho_r(u_{2n+1}, u_{2n+2}), \frac{(\rho_r(u_{2n}, u_{2n+2}) + \rho_r(u_{2n+1}, u_{2n+1}))}{2} \right\}$$

$= h\, \rho_r(u_{2n}, u_{2n+1})$.

Therefore, $\rho_r(u_{2n+1}, u_{2n+2}) \leq h\, \rho_r(u_{2n}, u_{2n+1})$.

Similarly $\rho_r(u_n, u_{2n+1}) \leq h\, \rho_r(u_{2n-1}, u_{2n})$ which implies $\rho_r(u_{2n+1}, u_{2n+2}) \leq h^2\, \rho_r(u_{2n-1}, u_{2n})$.

Proceeding in the same manner, we have

$\rho_r(u_{2n+1}, u_{2n+2}) \leq h^n\, \rho_r(u_0, u_1)$ and as $0 < h < 1$, which gives

$h^n \to 0$ as $n \to \infty$.

Thus $\{u_n\}$ is a Cauchy sequence in a complete rectangular metric-like space U. So by completeness of U, there exists a point $z \in U$ such that $u_n \to z$.

$\rho_r(Sz, Tu_{2n}) \leq h$

$$\max\left\{ \rho_r(z, u_{2n}), \rho_r(z, Sz), \rho_r(u_{2n}, Tu_{2n}), \frac{(\rho_r(z, Tu_{2n}) + \rho_r(u_{2n}, Sz))}{2} \right\}$$

$\rho_r(Sz, x_{2n+1}) \leq h$ max

$$\left\{ \rho_r(z, u_{2n}), \rho_r(z, Sz), \rho_r(u_{2n}, u_{2n+1}), \frac{(\rho_r(z, u_{2n+1}) + \rho_r(u_{2n}, Sz))}{2} \right\}$$

Taking the limit as $n \to \infty$, we have

$$\rho_r(Sz, z) \leq h \max\left\{ \rho_r(z, z), \rho_r(z, Sz), \rho_r(z, z), \frac{(\rho_r(z, z) + \rho_r(z, Sz))}{2} \right\}$$

$$\leq h \max\left\{ 0, \rho_r(z, Sz), 0, \frac{(0 + \rho_r(z, Sz))}{2} \right\}$$

This yields

$\rho_r(Sz, z) \leq h\, \rho_r(Sz, z)$ which is a contradiction as $0 < h < 1$.

Hence $Sz = z$.

Similarly, we can show that Tz = z.

This gives that z is the common fixed point of S and T.

Case 2: For uniqueness, let w ($w \neq z$) be another common fixed point of S and T.

Then by (3.1.1), we have

$\rho_r(z, w) = \rho_r(Sz, Tz)$

$$\leq h \max \{ \rho_r(z, w), \rho_r(z, Sz), \rho_r(w, Tw), \frac{(\rho_r(z, Tw) + \rho_r(w, Sz))}{2} \}$$

$$\leq h \max \{ \rho_r(z, w), \rho_r(z, z), \rho_r(w, w), \frac{(\rho_r(z, w) + \rho_r(w, z))}{2} \}$$

This gives,

$\rho_r(z, w) \leq h\, \rho_r(z, w)$ which is a contradiction as $0 < h < 1$.

Therefore, z = w, i.e. S and T have unique common fixed point.

Hence the theorem.

3.2 Example:

Let U = [0, 1] and $\rho_r = |u - v|$ be rectangular metric-like on U.

Let S, T: [0, 1] → [0, 1] defined by

$$S(u) = \begin{cases} 0\ , & \text{when } 0 \leq u < \frac{1}{2} \\ \frac{1}{5}\ , & \text{when } \frac{1}{2} < u \leq 1 \end{cases}$$

and

$$T(u) = \begin{cases} 0\ , & \text{when } 0 \leq u < 1 \\ \frac{1}{4}\ , & \text{when } u = 1 \end{cases}$$

Here, we discuss different cases according to various possible values of u and v in eq. (3.1.1). From equation (3.1.1), we have

$$|Su - Tv| \leq h \max \{ |u - v|, |u - Su|, |v - Tv|, \frac{|u - Tv| + |v - Su|}{2} \} \quad \dots (3.1.2)$$

Case 1 : If $0 \leq u \leq \frac{1}{2}$ and $0 \leq v < 1$, then in eq. (3.1.2) we have

$$|0 - 0| \leq h \max \left\{ |u - v|, |u - 0|, |v - 0|, \frac{|u - 0| + |v - 0|}{2} \right\}$$

$$0 \leq h \max \left\{ |u - v|, u, v, \frac{u + v}{2} \right\}$$

Taking all possible values of u and v, we see that, here, all the conditions of theorem (3.1) hold.

Case 2 : If $0 \le u \le \frac{1}{2}$ and v = 1, then in eq. (3.1.2), we have

$$\left|0-\frac{1}{4}\right| \le h \max\left\{|u-1|,\ |u-0|,\ \left|1-\frac{1}{4}\right|,\ \frac{\left|u-\frac{1}{4}\right|+|1-0|}{2}\right\}$$

$$\frac{1}{4} \le h \max\left\{|u-1|,\ u\ ,\ \frac{3}{4}\ ,\ \frac{\left|u-\frac{1}{4}\right|+1}{2}\right\}$$

Taking all possible values for u, we see that, all the conditions of theorem (3.1) hold.

Case 3 : If $\frac{1}{2} < u \le 1$ and $0 \le v < 1$, then in eq. (3.1.2), we have

$$\left|\frac{1}{5}-0\right| \le h \max\left\{|u-v|,\ \left|u-\frac{1}{5}\right|,\ |v-0|,\ \frac{|u-0|+\left|v-\frac{1}{5}\right|}{2}\right\}$$

$$\frac{1}{5} \le h \max\left\{|u-v|,\ \left|u-\frac{1}{5}\right|,\ v\ ,\ \frac{u+\left|v-\frac{1}{5}\right|}{2}\right\}$$

Taking all possible values of u and v, we see that, all the conditions of theorem (3.1) hold.

Case 4 : If $\frac{1}{2} < u \le 1$ and v = 1, then in eq. (3.1.2), we have

$$\left|\frac{1}{5}-\frac{1}{4}\right| \le h \max\left\{|u-1|,\ \left|u-\frac{1}{5}\right|,\ \left|1-\frac{1}{4}\right|,\ \frac{\left|u-\frac{1}{4}\right|+\left|v-\frac{1}{5}\right|}{2}\right\}$$

$$\frac{1}{20} \le h \max\left\{|u-1|,\ \left|u-\frac{1}{5}\right|,\ \frac{3}{4}\ ,\ \frac{\left|u-\frac{1}{4}\right|+\left|v-\frac{1}{5}\right|}{2}\right\}$$

Taking all possible values of u and v, we see that, all the conditions of theorem (3.1) hold.

Hence, all the conditions of theorem (3.1) are satisfied in all possible cases.

So, S and T have a unique common fixed point, i.e., zero.

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