A Two-State Retrial Queue with Reneging Customers

Neelam Singla[1] and Sonia Kalra[2]

[1]Assistant Professor, Department of Statistics, Punjabi University Patiala – 147002,

E-mail: neelgagan2k3@yahoo.co.in

[2]Research Scholar (Corresponding author), Department of Statistics,

Punjabi University, Patiala – 147002, E-mail: soniakalra276@gmail.com

Abstract

In this paper, we present an analysis of a single server retrial queueing system with reneging customers. The primary calls follow Poisson distribution. The repeating calls also follow the Poisson distribution but with a different parameter. Service times are exponentially distributed. The time dependent probabilities of exact number of arrivals and exact number of departures at when the server is free or when the server is busy from the system are obtained by solving the difference - differential equations recursively. Performance measures have been derived and system efficiency is discussed through numerical results. Graphical illustrations demonstrate how various parameters influence the behavior of the system.

Keywords: Retrial, Arrivals, Departures, Queueing, Probability, Reneging.

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1. Introduction

Queueing systems with repeated attempts arise in many practical situations of telephone services, computers and communication systems. As per retrial phenomena features, a customer on finding all the servers busy upon arrival, may join the virtual group of blocked customers called orbit and this customer retries for service from the orbit after some random amount of time. The queueing system with retrial phenomena is called retrial queue. The detailed overviews of main results and bibliographical information about retrial queues are given in Falin and Templeton (1997), Atralejo (1999), Corral (2006) and Atralejo and Corral (2008). Classical models of retrial queues are considered as an alternative to the queueing models with losses to represent the feature of redialing in telephone services. Another important feature in telephone services is customer impatience.

Impatient customers are of great importance as a lost customer is a significant loss to the business. On arrival if a customer finds the expected waiting time in the system to be more than his available time, then the customer refuses to join the system. In this situation the system is said to be balked. The reluctance of a customer to continue in queue after joining and waiting in queue is termed as reneging. Many researchers have thoroughly studied the effect of balking and reneging on queueing systems. Retrial queues with reneging customers explain real world situations more precisely. The notion of customer impatience appears in the queueing theory for the first time in the work of Haight (1957). Haight (1959) studied a queue with reneging in which he discussed about how to make rational decision while waiting in a queue and probable effect of this decision etc. Ancker and Gafarian (1963) studied M/M/1/N queueing system with balking and reneging and perform its steady-state analysis. Obert (1979) analyzed single server queues with reneging. Medhi and Choudhary (2013) presented explicit results for the M/M/c /c queuing system assuming that customers are reneging type and few re-designed performance measures also represented. Mahdy and Nabwey (2011) analyzed the markovian model with balking, reneging and heterogeneous servers with finite capacity.

Pegden and Rosenshine (1982) provided a closed-form expression for the probability of exactly i arrivals and j departures over a time horizon of length t in a classical queueing model M/M/1/∞. This measure supplies better insight into the behavior of a queueing system than the probability of the exact number of units in the system at a given time studied in the early literature on queues and therefore this measure is more justified. Indra and Renu (2011) presented the time dependent probabilities of exact arrivals and departures by time t for M/M/1 queueing model with Bernoulli schedule and multiple working vacations. Garg and Kumar (2012) obtained explicit time dependent probabilities of exact number of arrivals and departures from the orbit of a single server retrial queue with impatient customers.

In this paper, we consider a single server retrial queueing system with reneging customers and obtain the time dependent probabilities for the exact number of arrivals and departures from the system by a given time t for when the server is busy and when the server is idle by solving the difference-differential equations recursively.

Rest of this paper is organized as follows: In Section 2, the mathematical model is described and the difference-differential equations are derived. In Section 3, the time dependent solution for the model is obtained. Section 4 presents some important performance measures and verification of results. In Section 5, graphical illustrations and numerical results are presented to study the influence of the arrival rate, reneging rate and retrial rate on various probabilities. In the last Section 6, the busy period distribution of the system and the busy period distribution of the server is obtained numerically and presented graphically and at last the paper ends with a suitable conclusion.

2. Model Description

We consider a two-state M/M/1 retrial queueing system with reneging customers. A primary customer, who enters the queueing system, arrives according to a Poisson process with parameter λ and receives service immediately if the server is idle otherwise a customer joins a pool of waiting customers called orbit. A secondary customer in the orbit repeats his request for service following Poisson distribution with retrial rate θ. Service time follows exponential distribution with service rate μ. The customer in queue may become impatient when the service is not available for a long time, then the customer decides to abandon the queue with probability (1- α) or decides to join the orbit with probability α. For distribution of arrivals, service times and retrials we use the following assumptions and notations:

i. The arrival of primary calls follow a Poisson distribution with parameter λ.
ii. The repeated calls follow a Poisson distribution with parameter θ.
iii. Service times are exponentially distributed with parameter μ.
iv. The stochastic process involved viz. arrivals of units, departures of units and retrials are statistically independent.

Laplace transformation $\bar{f}$(s) of $f(t)$ is given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \text{Re}(s) > 0$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} \; is \; \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k-l}\, e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)} \, (p-a_k)^{m_k} \;\; \forall \; p = a_k, \;\; a_i \neq a_k \text{ for } i \neq k.$$

where,

$$P(p) = (p-a_1)^{m_1} \, (p-a_2)^{m_2} \ldots\ldots\ldots (p-a_n)^{m_n}$$

Q(p) is a polynomial of degree < $m_1+m_2+m_3+\ldots\ldots\ldots\ldots m_n-1$.

If L^{-1}{f(s)} = F(t) and L^{-1}{g(s)} = G(t), then

$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)G(t-u)du = F * G$, F * G is called the convolution of F and G.

2.1 The Two-Dimensional State Model

Definitions

$P_{i,j,0}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is idle.

$P_{i,j,1}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is busy.

$P_{i,j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \qquad \forall\ i, j \qquad i \geq j$$

also

$$P_{i,j,1}(t) = 0,\ i \leq j;\ P_{i,j,0}(t) = 0,\ i < j.$$

Initially

$$P_{0,0,0}(0) = 1;\quad P_{i,j,0}(0) = 0\ \&\ P_{i,j,1}(0) = 0,\ i, j \neq 0.$$

The difference – differential equations governing the system are

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu P_{i,j-1,1}(t) \qquad i \geq j \geq 0 \qquad \textbf{(1)}$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) \qquad \textbf{(2)}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda + \mu + (i-j-1)\theta(1-\alpha))P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda(1-\delta_{i-1,j})P_{i-1,j,1}(t) +$$

$$(i-j)\theta P_{i,j,0}(t) + (i-j)\theta(1-\alpha)P_{i,j-1,1}(t) \qquad i>1,\ i > j \geq 0 \qquad \textbf{(3)}$$

$$\text{where } \delta_{i-1,j} = \begin{cases} 1, & when\ i-1 = j \\ 0, & otherwise \end{cases}$$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad Re(s) > 0$$

in the equations (1) - (3) along with the initial conditions, we have

$$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu \bar{P}_{i,j-1,1}(s) \qquad i \geq j \geq 0 \qquad \textbf{(4)}$$

$$(s + \lambda + \mu)\bar{P}_{1,0,1}(s) = \lambda \bar{P}_{0,0,0}(s) \qquad \textbf{(5)}$$

$$(s+\lambda+\mu+(i-j-1)\theta(1-\alpha))\bar{P}_{i,j,1}(s) = \lambda\bar{P}_{i-1,j,0}(s) + \lambda(1-\delta_{i-1,j})\bar{P}_{i-1,j,1}(s) + (i-j)\theta\bar{P}_{i,j,0}(s) + (i-j)\theta(1-\alpha)\bar{P}_{i,j-1,1}(s) \qquad i>1,\ i>j\geq 0 \tag{6}$$

$$\text{where } \delta_{i-1,j} = \begin{cases} 1, & when\ i-1=j \\ 0, & otherwise \end{cases}$$

3. Solution of the Problem

Solving equations (4) to (6) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \tag{7}$$

$$\bar{P}_{1,1,0}(s) = \frac{\lambda\mu}{(s+\lambda)^2(s+\lambda+\mu)} \tag{8}$$

$$\bar{P}_{1,0,1}(s) = \frac{\lambda}{(s+\lambda)(s+\lambda+\mu)} \tag{9}$$

$$\bar{P}_{i,1,0}(s) = \frac{\mu}{s+\lambda+(i-1)\theta}\bar{P}_{i,0,1}(s) \qquad i>1 \tag{10}$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu}{s+\lambda}\left(\frac{\lambda}{(s+\lambda+\mu)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{(s+\lambda+\mu)}\bar{P}_{i,i-1,0}(s) + \frac{\theta(1-\alpha)}{(s+\lambda+\mu)}\bar{P}_{i,i-2,1}(s)\right) \qquad i>1 \tag{11}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s+\lambda+(i-j)\theta}\left[\sum_{k=1}^{i-j+1}\left\{\begin{matrix}\prod_{m=k}^{i-j}\left(\frac{1}{(s+\lambda+\mu+m\theta(1-\alpha))}\right)^{\varphi'_k(s)}(\lambda)^{\varphi'_k(s)} \\ (\lambda)^{(i-j-k)\varphi'_k(s)}\eta'_k(s)\bar{P}_{j+k-1,j-1,0}(s)\end{matrix}\right\} + \sum_{k=1}^{i-j}\left\{\prod_{m=k}^{i-j}(k+1)\theta(1-\alpha)(\lambda)^{(i-j-k)}\frac{1}{(s+\lambda+\mu+m\theta(1-\alpha))}\bar{P}_{j+k,j-2,1}(s)\right\} + \prod_{p=1}^{i-j}\left\{\frac{1}{(s+\lambda+\mu+p\theta(1-\alpha))}(\lambda)^{(i-j)}\bar{P}_{j,j-1,1}(s)\right\}\right] \qquad i>j>1 \tag{12}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 \ldots\ldots\ldots\ldots\ldots\ldots.. & for\ k=1 \\ \left(1+\frac{k\theta}{(s+\lambda+\mu+(k-1)\theta(1-\alpha))}\right) & for\ k=2\ to\ i-j \\ \frac{k\theta}{(s+\lambda+\mu+(k-1)\theta(1-\alpha))} & for\ k=i-j+1 \end{cases}$$

$$\text{and} \quad \varphi'_k(s) = \begin{cases} 1 & \text{for } k = 1 \text{ to } i-j-1 \\ 0 & \text{for } \quad k = i-j \end{cases}$$

$$\bar{P}_{i,0,1}(s) = \prod_{m=1}^{i-1}\frac{(\lambda)^{i-1}}{(s+\lambda+\mu+m\theta(1-\alpha))} * \bar{P}_{1,0,1}(s) \qquad i>1 \tag{13}$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s+\lambda+\mu}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu}\bar{P}_{i,i-1,0}(s) + \frac{\theta(1-\alpha)}{(s+\lambda+\mu)}\bar{P}_{i,i-2,1}(s) \qquad i>1 \tag{14}$$

$$\bar{P}_{i,j,1}(s) = \left[\sum_{k=1}^{i-j}\left\{\begin{matrix}\prod_{m=k}^{i-j-1}\left(\frac{1}{(s+\lambda+\mu+m\theta(1-\alpha))}\right)^{\varphi'_k(s)}(\lambda)^{\varphi'_k(s)} \\ (\lambda)^{(i-j-k-1)\varphi'_k(s)}\eta'_k(s)\bar{P}_{j+k,j,0}(s)\end{matrix}\right\} + \sum_{k=1}^{i-j-1}\left\{\prod_{m=k}^{i-j-1}(k+1)\theta(1-\alpha)(\lambda)^{(i-j-k-1)}\frac{1}{(s+\lambda+\mu+m\theta(1-\alpha))}\bar{P}_{j+k+1,j-1,1}(s)\right\} + \prod_{p=1}^{i-j-1}\left\{\frac{1}{(s+\lambda+\mu+p\theta(1-\alpha))}(\lambda)^{(i-j-1)}\bar{P}_{j,j-1,1}(s)\right\}\right]$$

$i \geq j+2,\ j \geq 1$ **(15)**

$$\text{where } \eta'_k(s) = \begin{cases} 1 \ldots\ldots\ldots\ldots\ldots & for\ k=1 \\ \left(1+\frac{k\theta}{(s+\lambda+\mu+(k-1)\theta(1-\alpha))}\right) & for\ k=2\ to\ i-j-1 \\ \frac{k\theta}{(s+\lambda+\mu+(k-1)\theta(1-\alpha))} & for\ k=i-j \end{cases}$$

$$\text{and}\quad \varphi'_k(s) = \begin{cases} 1 & \text{for k} = 1 \text{ to i} - \text{j} - 1 \\ 0 & \text{for} \quad \text{k} = \text{i} - \text{j} \end{cases}$$

Taking the Inverse Laplace transform of equations (7) to (15), we have

$$P_{0,0,0}(t) = e^{-\lambda t} \quad \textbf{(16)}$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left\{\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} \quad \textbf{(17)}$$

$$P_{i,1,0}(t) = \mu e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) \qquad i \geq 1 \quad \textbf{(18)}$$

$$P_{i,i,0}(t)= \left[(\lambda\mu)e^{-\lambda t}\left\{\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P_{i-1,i-1,0}(t) + (\mu\theta)e^{-\lambda t}\left\{\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P_{i,i-1,0}(t) + (\mu\theta)(1-\alpha)e^{-\lambda t}\left\{\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P_{i,i-2,1}(t)\right] \qquad i > 1 \quad \textbf{(19)}$$

$$\begin{aligned} P_{i,j,0}(t) = {} & \mu\,\lambda^{i-j}\, e^{-(\lambda+(i-j)\theta)t}\left\{\prod_{m=1}^{i-j}\frac{1}{(\mu+m\theta(1-\alpha))^m} - e^{-(\mu+m\theta(1-\alpha))t}\sum_{r=0}^{m-1}\frac{t^r}{r!}\frac{1}{(\mu+m\theta(1-\alpha))^{m-r}}\right\} * P_{j,j-1,0}(t) + {} \\ & \lambda\mu e^{-(\lambda+(i-j)\theta)t}\left[\sum_{k=2}^{i-j}(\lambda)^{i-j-k}\left\{\prod_{m=k}^{i-j}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+1}} - e^{-(\mu+m\theta(1-\alpha))t}\sum_{r=0}^{m-k}\frac{t^r}{r!}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+1-r}}\right\} * P_{j+k-1,j-1,0}(t)\right] + {} \\ & \lambda\mu e^{-(\lambda+(i-j)\theta)t}\left[\sum_{k=2}^{i-j}(\lambda)^{i-j-k}(k\theta)\left\{\prod_{m=k-1}^{i-j}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+2}} - e^{-(\mu+m\theta(1-\alpha))t}\sum_{r=0}^{m-k+1}\frac{t^r}{r!}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+2-r}}\right\} * P_{j+k-1,j-1,0}(t)\right] + {} \\ & \mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\mu+(i-j)\theta(1-\alpha)} - \frac{e^{-(\mu+(i-j)\theta(1-\alpha))t}}{\mu+(i-j)\theta(1-\alpha)}\right\} * P_{i,j-1,0}(t) + {} \\ & \mu e^{-(\lambda+(i-j)\theta)t}\left[\sum_{k=1}^{i-j}(\lambda)^{i-j-k}(k+1)\theta(1-\alpha)\left\{\prod_{m=k}^{i-j}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+1}} - e^{-(\mu+m\theta(1-\alpha))t}\sum_{r=0}^{m-k}\frac{t^r}{r!}\frac{1}{(\mu+m\theta(1-\alpha))^{m-k+1-r}}\right\} * P_{j+k,j-2,1}(t)\right] + {} \\ & \mu(\lambda)^{i-j}e^{-(\lambda+(i-j)\theta)t}\left\{\prod_{p=1}^{i-j}\frac{1}{(\mu+p\theta(1-\alpha))^p} - e^{-(\mu+p\theta(1-\alpha))t}\sum_{r=0}^{p-1}\frac{t^r}{r!}\frac{1}{(\mu+p\theta(1-\alpha))^{p-r}}\right\} * P_{j,j-1,1}(t)\end{aligned}$$

$i>j>1$ **(20)**

$$P_{i,0,1}(t) = (\lambda)^{i-1}\{\prod_{m=1}^{i-1} e^{-(\lambda+\mu+m\theta(1-\alpha))t}\} * P_{1,0,1}(t) \qquad i > 1 \qquad \textbf{(21)}$$

$$P_{i,i-1,1}(t) = \left(\lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + \theta(1-\alpha)e^{-(\lambda+\mu)t} * P_{i,i-2,1}(t)\right) \qquad i > 1 \qquad \textbf{(22)}$$

$$P_{i,j,1}(t) = \lambda^{i-j-1}\{\prod_{m=1}^{i-j-1} e^{-(\lambda+\mu+m\theta(1-\alpha))t}\}\frac{t^{m-1}}{(m-1)!} * P_{j+1,j,0}(t) + \sum_{k=2}^{i-j-1}\left[\lambda^{i-j-k}\{\prod_{m=k}^{i-j-1} e^{-(\lambda+\mu+m\theta(1-\alpha))t}\}\frac{t^{m-k}}{(m-k)!} * P_{j+k,j,0}(t)\right] + \sum_{k=2}^{i-j-1}\left[\lambda^{i-j-k}(k\theta)\{\prod_{m=k-1}^{i-j-1} e^{-(\lambda+\mu+m\theta(1-\alpha))t}\}\frac{t^{m-k}}{(m-k)!} * P_{j+k,j,0}(t)\right] + (i-j)\,\theta e^{-(\lambda+\mu+(i-j-1)\theta(1-\alpha))t} * P_{i,j,0}(t) + \sum_{k=1}^{i-j-1}\left[(\lambda)^{i-j-k-1}(k+1)\theta\,(1-\alpha)\{\prod_{m=k}^{i-j-1} e^{-(\lambda+\mu+m\theta(1-\alpha))t}\}\frac{t^{m-k}}{(m-k)!} * P_{j+k+1,j-1,1}(t)\right] + \left[(\lambda)^{i-j-1}\{\prod_{p=1}^{i-j-1} e^{-(\lambda+\mu+p\theta(1-\alpha))t}\}\frac{t^{p-1}}{(p-1)!} * P_{j+1,j,1}(t)\right]$$

$$i \geq j+2,\ j \geq 1 \qquad \textbf{(23)}$$

4. Some Important Performance Measures

4.1 The Laplace transform of the probability $P_{i.}(t)$ that exactly i units arrive by time t is :

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}; \qquad i > 0 \qquad \textbf{(24)}$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}. \qquad \textbf{(25)}$$

The very (basic) assumption on primary arrivals is that it forms a Poisson process and above analysis of abstract solution also verifies the same.

4.2 The probability that exactly j customers have been served by time t, $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

4.3 From the abstract solution of our model, we verified that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (7) - (15) and adding, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s)\} = \frac{1}{s}.$$

After taking the inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t) + P_{i,j,1}(t)\} = 1.$$

which is a verification of our results.

4.4 Define $Q_{n,k}(t)$ as the probability that there are n customers in the system at time t and the server is free or busy according as k=0 or 1.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}(t)$ and $P_{i,j,1}(t)$:

When the server is free, it is defined by probability $Q_{n,0}(t)$:

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures).

When the server is busy, it is defined by probability $Q_{n,1}(t)$:

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures – 1).

Using above definitions, in the equations (1) to (3), the set of equations in statistical equilibrium are:

$$(\lambda + n\theta)\, Q_{n,0} = \mu\, Q_{n,1} \qquad n \geq 0 \qquad \textbf{(26)}$$

$$(\lambda + \mu + n\theta(1-\alpha))\, Q_{n,1} = \lambda Q_{n,0} + \lambda Q_{n-1,1} + (n+1)\,\theta\, Q_{n+1,0} + (n+1)\theta(1-\alpha) Q_{n+1,1} \qquad n > 1 \qquad \textbf{(27)}$$

when we consider α=1, then the above set of equations are

$$(\lambda + n\theta)\, Q_{n,0} = \mu\, Q_{n,1} \qquad n \geq 0 \qquad \textbf{(28)}$$

$$(\lambda + \mu) Q_{n,1} = \lambda Q_{n,0} + \lambda Q_{n-1,1} + (n+1)\,\theta\, Q_{n+1,0} \qquad n>1 \qquad \textbf{(29)}$$

These equations coincide with the equations (1.5) and (1.6) of "Falin and Templeton (1997)".

5. Numerical Solution and Graphical Representation

The numerical results for specific values of parameters are evaluated using MATLAB programming. The effect of traffic intensity $\rho = \left(\frac{\lambda}{\mu}\right)$ and reneging parameter (1-α) is studied on the performance of our system. From the numerical results, it is found that the sum of all the probabilities at any instance for the case ρ=0.3, $\eta \left(\frac{\theta}{\mu}\right)$ =0.6, (1-α) = 0.3 approaches to one. In table 1, we show some of the significant probabilities at different instant of time and their sum.

Table-1: Some probabilities of system state at different instants of time

At time t= 1

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	Sum
0.7408	0.0818	0.0057	0.1405	0.0167	0.0086	0.9941

At time t= 5

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{1,0,1}$	$P_{2,0,1}$
0.2231	0.2682	0.0226	0.1369	0.0210	0.0403	0.0085	0.0079	0.0665	0.0165

$P_{2,1,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,2,1}$	$P_{4,3,1}$	**Sum**
0.0750	0.0225	0.0355	0.0121	0.0098	0.9664

At time t= 10

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,4,0}$	$P_{5,5,0}$
0.0498	0.1344	0.0063	0.1738	0.0161	0.1430	0.0196	0.0842	0.00151	0.0379

$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,2,0}$	$P_{7,7,0}$	$P_{1,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,1,1}$	$P_{4,2,1}$	$P_{5,3,1}$
0.0083	0.0137	0.0090	0.0041	0.0149	0.0402	0.0118	0.0517	0.0170	0.0421	0.0142

$P_{5,4,1}$	$P_{6,4,1}$	$P_{6,3,1}$	$P_{6,5,1}$	**Sum**
0.0190	0.0077	0.0085	0.0121	0.9496

At time t= 30

$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{8,7,0}$	$P_{8,8,0}$
0.0047	0.0134	0.0287	0.0491	0.0035	0.0697	0.0060	0.0844	0.0084	0.0890

$P_{9,8,0}$	$P_{9,9,0}$	$P_{10,8,0}$	$P_{10,9,0}$	$P_{10,10,0}$	$P_{4,3,1}$	$P_{5,4,1}$	$P_{6,4,1}$	$P_{6,5,1}$	$P_{7,5,1}$
0.0101	0.0831	0.0025	0.0219	0.3320	0.0040	0.0086	0.0026	0.0147	0.0046

$P_{7,6,1}$	$P_{8,6,1}$	$P_{8,7,1}$	$P_{9,7,1}$	$P_{9,8,1}$	$P_{10,7,1}$	$P_{10,8,1}$	$P_{10,9,1}$	**Sum**
0.0209	0.0068	0.0253	0.0086	0.0267	0.0028	0.0130	0.0396	0.9837

At time t= 40

$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{9,8,0}$	$P_{9,9,0}$	$P_{10,9,0}$	$P_{10,10,0}$
0.0047	0.0110	0.0211	0.0348	0.0026	0.0501	0.0042	0.0639	0.0132	0.7052

$P_{6,5,1}$	$P_{7,6,1}$	$P_{8,7,1}$	$P_{9,7,1}$	$P_{9,8,1}$	$P_{10,8,1}$	$P_{10,9,1}$	**Sum**
0.0033	0.0063	0.0104	0.0033	0.0150	0.0068	0.0303	0.9862

Various probabilities are plotted against time t through figures 1 to 6. Figure 1 shows the plot of probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against time t for ρ=0.3, η=0.6 and (1-α) = 0.3. It is clear from the graph that the probability $P_{0,0,0}$ decreases rapidly from the initial value =1 at time t=0. The probability $P_{1,1,0}$ increases in the starting moments, from the initial value zero at time t=0 and then decreases gradually.

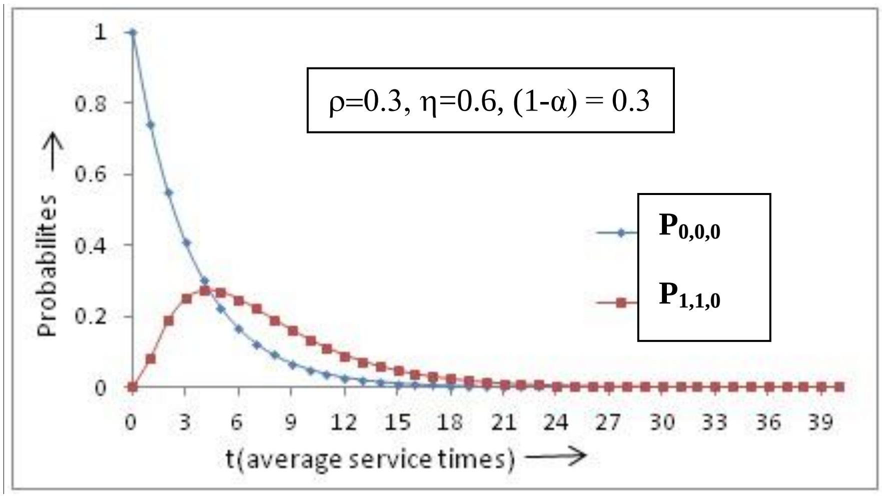

Figure 1: Probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against time t

Figure 2 shows the comparison among the probabilities $P_{5,1,0}$ and $P_{5,1,1}$ for the case when ρ=0.6, η=0.9 and (1- α) =0.2. In the starting moments, the probabilities increase, but afterwards probabilities start decreasing. From the graph, it can also be interpreted that the probability $P_{5,1,1}$ (server busy) is always more than the probability $P_{5,1,0}$ (server free).

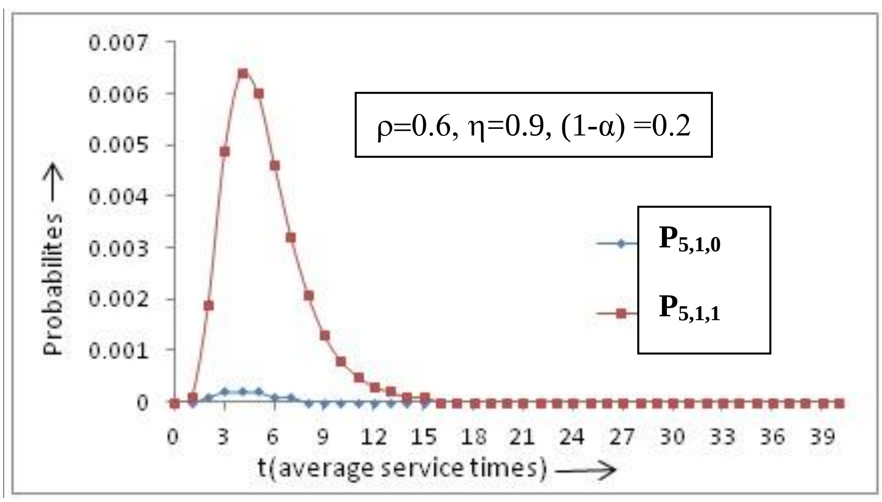

Figure 2: Probabilities $P_{5,1,0}$ and $P_{5,1,1}$ vs. time t

Figure 3 depicts the plot of probabilities $P_{3,1,1}$, $P_{4,1,1}$, $P_{5,1,1}$ and $P_{6,1,1}$ against time t for the case when ρ=0.6, η=0.9 and (1-α) =0.2. It is seen that the probabilities increases in the starting moments, then decreases with high rate. From the graph, it is also observed that probabilities have smaller values when i (number of arrivals) are more.

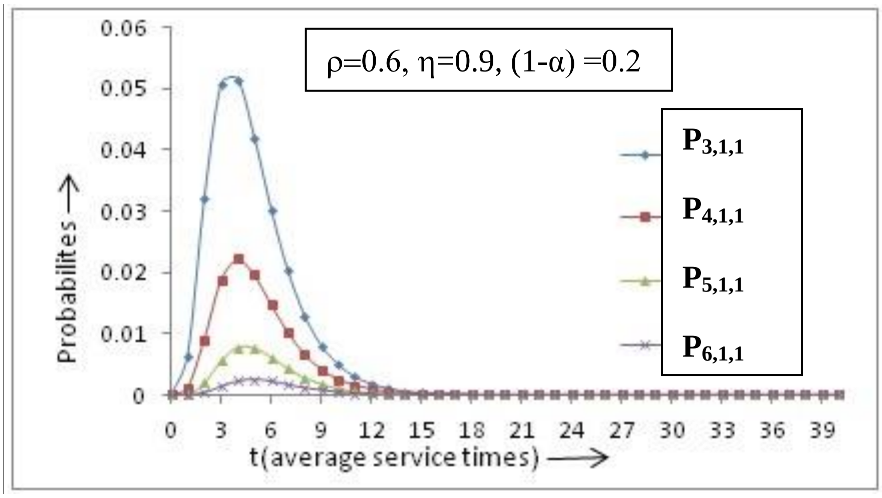

Figure 3: Probabilities $P_{3,1,1}$, $P_{4,1,1}$, $P_{5,1,1}$ and $P_{6,1,1}$ against time t

Figure 4 illustrates the relationships among four probabilities with different number of departures and when number of arrivals is same against time t for the case when ρ=0.3, η=0.6 and (1-α) =0.3. The probabilities increase rapidly in the starting moments and then decrease for higher values of t. Here, it is also noticed that in general higher the number of departures, higher is the corresponding probability $P_{4,3,1} > P_{4,2,1} > P_{4,1,1} > P_{4,0,1}$.

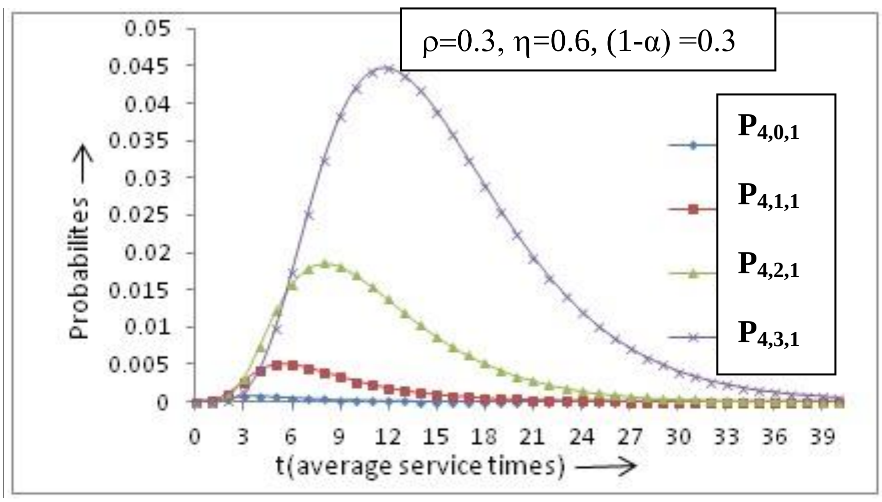

Figure 4: Probabilities $P_{4,0,1}$, $P_{4,1,1}$, $P_{4,2,1}$ and $P_{4,3,1}$ against time t

To study the effect of changing ρ on different probabilities of the model, the data of various probabilities is generated for different values of ρ keeping the other parameters constant. The set of ρ values for the comparison is taking {0.3,0.6,0.9}. Figure 5 shows plot of probability $P_{0,0,0}$ against time t for different values of ρ. From the graph, it is concluded that as ρ increases $P_{0,0,0}$ decreases. So more the traffic intensity i.e. when more customers are arriving per unit service time, less is the probability of zero units in the system.

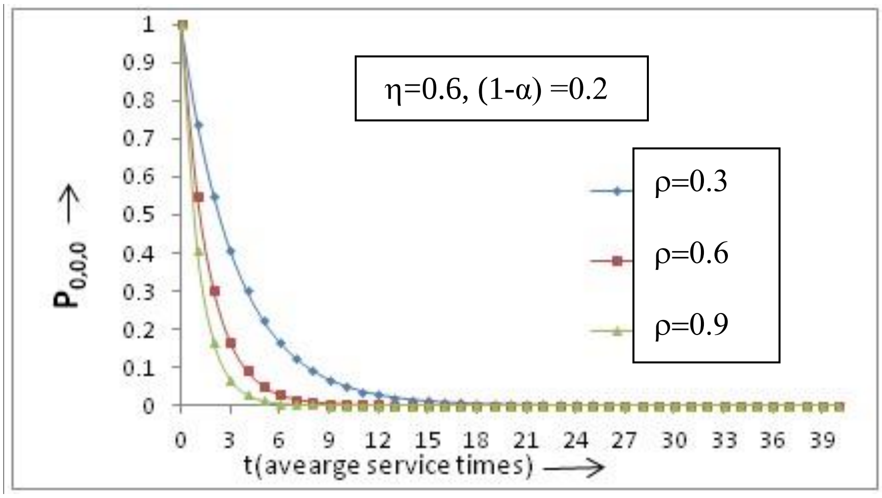

Figure 5: Effect of $\rho=[\lambda/\mu]$ on $P_{0,0,0}$ against time

The probability $P_{3,1,1}$ is plotted in figure 6 to study the behavior of reneging parameter by generating data for different values of (1-α). We can see from the figure, as the value of reneging parameter increases, the probability of server being busy decreases. In the starting moments, the probabilities increase but afterwards start decreasing due to their peculiar behavior.

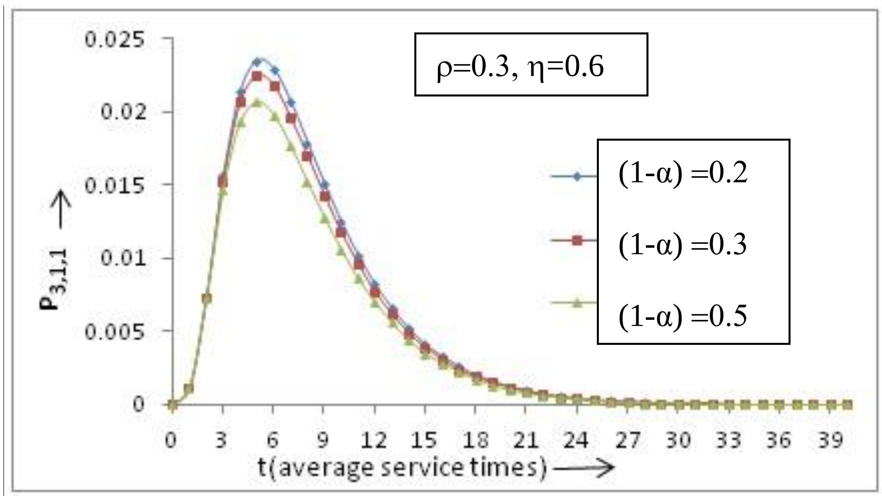

Figure 6: Effect of (1-α) on $P_{3,1,1}$ against time

6. Busy Period Distribution

We discuss some interesting numerical results about busy period distribution of the server and busy period distribution of the system.

The probability when the server is busy is given by

$$P\text{ (Server is busy)} = \sum_{i>j\geq 0} P_{i,j,1}(t) \tag{30}$$

The probability when the system is busy is given by

$$P\text{ (System is busy)} = \sum_{i>j\geq 0}\left(P_{i,j,0}(t) + P_{i,j,1}(t)\right) \quad \textbf{(31)}$$

Following the work of Bunday (1986), the numerical results are generated using MATLAB programming. The probability when the system is busy and the probability when the server is busy are presented in Table-2 for different values of ρ (traffic intensity).

Table-2: Probability of system busy and server busy for different values of ρ.

(η=0.6, (1-α) = 0.2)

	Probability (System busy)			Probability (Server busy)		
t	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.1751	0.3232	0.4772	0.1688	0.3024	0.4087
2	0.2436	0.4457	0.6034	0.2199	0.3783	0.4943
3	0.282	0.5189	0.6927	0.2408	0.4115	0.5335
4	0.3078	0.5694	0.7507	0.2527	0.4332	0.5604
5	0.3261	0.606	0.7904	0.2607	0.4493	0.5802
6	0.3396	0.6331	0.8186	0.2665	0.4616	0.5942
7	0.3495	0.6537	0.8389	0.2709	0.471	0.6018

Figures 7 and 8 show the relationships between the probability when the system is busy and the probability when the server is busy for different values of ρ i.e. ρ=0.3,0.6,0.9. From these figures, it is apparent that the probability achieved higher peak values for higher value of ρ.

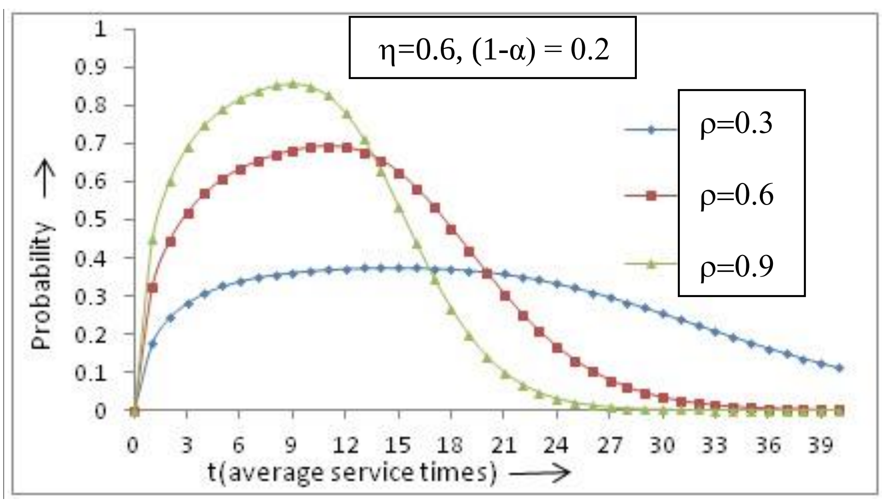

Figure 7: Effect of ρ on Probability (system busy) against time

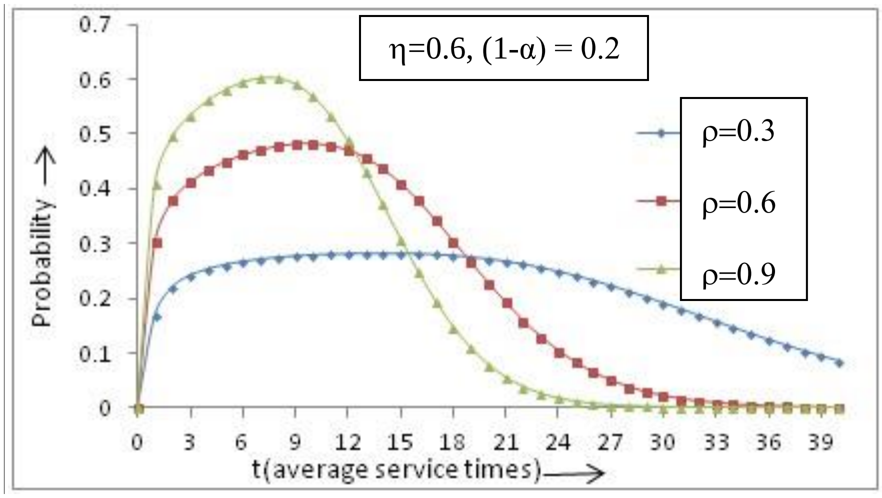

Figure 8: Effect of ρ on Probability (server busy) against time

In figure 9, the probability (system busy) and the probability (server busy) are plotted for ρ=0.9, η=0.6 and (1-α) = 0.2. It is observed that the two probabilities are increasing rapidly in starting moments and start decreasing gradually for higher values of t. From these figures, it also can also be seen that the probability when the system is busy always remain more than the probability when the server is busy.

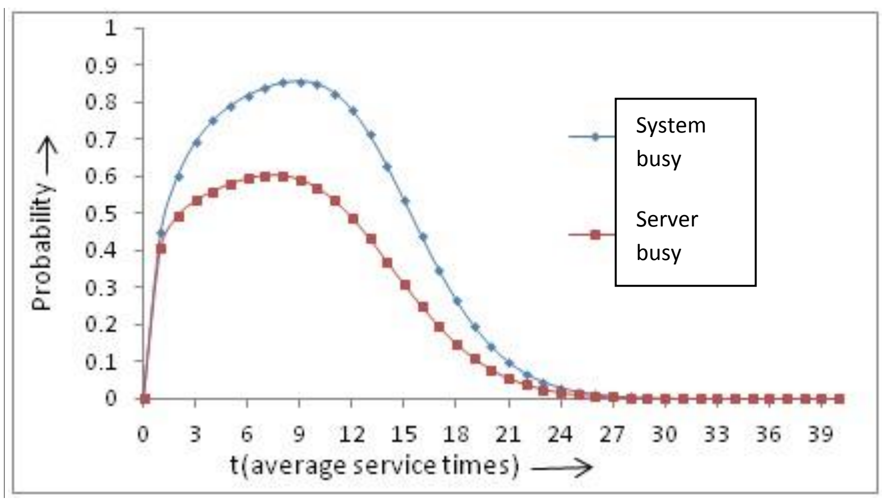

Figure 9: Probability (system busy) and Probability (server busy) against time for ρ=0.9, η=0.6, (1-α) = 0.2

7. Conclusion

Managing customer impatience plays a vital role in improving the efficiency of queueing systems. Reneging of impatient customers leads to loss of potential customers, which results in the loss of business. In this paper, we analyzed a single server retrial queueing system with reneging customers. This paper focuses on to find the time dependent probabilities of exact numbers of arrivals in the system and exact numbers of departures from the system. Various performance measures and verification of results are computed. Numerical results have been generated using MATLAB programming to investigate the effects of several parameters on the system.

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