A New α-Cut Approach to Solve Fuzzy Game Problem

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ABSTRACT

The paper considers a two person zero sum game in which fuzzy payoffs are pentagonal fuzzy numbers. A ranking method is proposed to convert pentagonal fuzzy numbers into crisp number and it is used to solve fuzzy game problems. The fuzzy game problem is converted into crisp problem and then solved by using traditional method. The proposed method is illustrated with the help of numerical examples.

Keywords: *Pentagonal fuzzy numbers, Ranking method, zero sum game, fuzzy game theory.*

1. Introduction

Game theory owes its origin to Mathematician John von Neumann and Economist Oskar Morgenstren (1947). It deals with decision making in situations of conflict and cooperation between intelligent and rational decision-makers. In a game problem each player attempts to take best decision by selecting various strategies from the set of available strategies. Game theory plays an important role in decision making fields such as defence, economics, political science, management etc. John Nash (1949) proved that a finite game problem always has an equilibrium point at which all players select their best actions, when the opponent's choices are given.

The traditional game theory assumes the existence of exact payoffs to solve competitive situations. However in the real life game situations such precise information on the payoffs is not available. Due to lack of information, the players are not able to estimate exactly payoffs in real situations. This lack of certainty may be appropriately modelled by using fuzzy set.

The concept of fuzzy set theory deals with imprecision, vagueness in real life situations. It was firstly proposed by Zadeh (1965). Bellman and Zadeh (1970) elaborated on the concept of decision making in the fuzzy environment. Later on, fuzzy methodologies have been successfully applied in a wide range of real world situations. Jain (1976) was the first to propose method of ranking fuzzy numbers for decision making in fuzzy situations. Dhanalaxmi and Kennedy (2014) proposed some ranking methods for octagonal fuzzy numbers. Lee and Yun (2014) define extension of zadeh's principle for pentagonal fuzzy numbers and conclude a triangular fuzzy numbers after the addition and subtraction of numbers. Namarta and Neha (2016) proposed ranking of hendecagonal fuzzy numbers by using centroid of centroids. Selvakumari and Lavanya (2015) proposed a method for all fuzzy game problems where entries of payoff matrix are trapezoidal and triangular fuzzy number. Kamble (2017) discussed pentagonal fuzzy numbers by using α-cut approach. Thirucheran et al. (2017) considered a two person zero sum game whose imprecise values are triangular fuzzy numbers and solve it without converting into crisp problem using ranking criteria.

The present paper describes the method of ranking pentagonal fuzzy numbers by using α-cut technique. In this paper, the method of ranking fuzzy numbers is applied on fuzzy game problems. By using this technique fuzzy game problem is converted into crisp problem and then solved by arithmetic method.

The paper is organized into seven sections. In Section 2, some basic definitions of fuzzy set and pentagonal fuzzy numbers are given. Section 3 presents proposed ranking method of fuzzy numbers. In Section 4, mathematical formulation of fuzzy game problem is described. Section 5, describes an Arithmetic method to solve game problem. In Section 6, proposed method is illustrated with the help of numerical examples. Finally conclusion is presented in Section 7.

2. Pentagonal Fuzzy Numbers:

Definition2.1. A generalised fuzzy number $\widetilde{A_H}$ =$(a_1, a_2, a_3, a_4, a_5;\ w)$ is said to be pentagonal fuzzy number if its membership function $\mu_{\widetilde{A_p}}$(x) is given below:

$$\mu_{\widetilde{A_H}}(x) = \begin{Bmatrix} 0 & x \le a_1 \\ \frac{w}{2}\left(\frac{x-a_1}{a_2-a_1}\right) & a_1 \le x \le a_2 \\ \frac{w}{2} + \frac{w}{2}\left(\frac{x-a_2}{a_3-a_2}\right) & a_2 \le x \le a_3 \\ w & x = a_3 \\ w - \frac{w}{2}\left(\frac{x-a_3}{a_4-a_3}\right) & a_3 \le x \le a_4 \\ \frac{w}{2}\left(\frac{a_5-x}{a_5-a_4}\right) & a_4 \le x \le a_5 \\ 0 & x \ge a_5 \end{Bmatrix}$$

Definition2.2. The α- cut of a normal heptagonal fuzzy number $\widetilde{A_H} = (a_1, a_2, a_3, a_4, a_5\ )$ is

$$\tilde{A} = [A_\alpha{}^L, A_\alpha{}^R] = \begin{Bmatrix} \left(A_\alpha{}^L\right)_1, \left(A_\alpha{}^R\right)_1 & \alpha \in [0, k] \\ \left(A_\alpha{}^L\right)_2, \left(A_\alpha{}^R\right)_2 & \alpha \in [k, 1] \end{Bmatrix}$$

Where $\left(A_\alpha{}^L\right)_1 = a_1 + \frac{r}{k}(a_2 - a_1)$, $\left(A_\alpha{}^L\right)_2 = a_2 + \frac{1-s}{1-k}(a_3 - a_2)$,

$$\left(A_\alpha{}^R\right)_1 = a_{4-} \frac{1-s}{1-k}(a_3 - a_2)\ , \left(A_\alpha{}^R\right)_2 = a_{\ 5} - \frac{r}{k}(a_5 - a_4)$$

3. Proposed Ranking Technique:

$$R(\tilde{A}_H) = \frac{1}{2}\int_0^k \{l_1(r) + l_2(r)\}dr + \frac{1}{2}\int_k^1 \{s_1(t) + s_2(t)\}ds$$

$$R(\tilde{A}_H) = \frac{1}{2}\int_0^k [a_1 + \frac{r}{k}(a_2 - a_1) + a_{5-} \frac{r}{k}(a_5 - a_4)]dr + \frac{1}{2}\int_k^1 [a_2 + \frac{1-s}{1-k}(a_3 - a_3) + a_{\ 4} - \frac{1-s}{1-k}(a_4 - a_3)]ds$$

On solving,

$$\text{R}(\tilde{A}_H) = \frac{1}{2}\{(a_1 + 2a_3 + 2a_4 + a_5)k + (a_2 - 2a_3 + a_4)\}$$

4. Mathematical Formulation of Fuzzy Game problem:

Consider a two person zero sum fuzzy game in which all the entries in the payoffs matrix are octagonal fuzzy numbers. Let player A has 'm' strategies and player B has 'n' strategies. Here it is assumed that each player has to choose strategy from the pure strategies. Player A is always assumed to be gainer and player B is always looser. The payoff matrix m $\times n$ is

$$A = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & .... & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & .... & \tilde{a}_{2n} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & .... & \tilde{a}_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \tilde{a}_{m3} & .... & \tilde{a}_{mn} \end{bmatrix}$$

5. Arithmetic (oddment) Method for n × n Games:

When a game matrix cannot be reduced in size by using the principle of dominance, then an easy method to solve game problem is arithmetic method as follows:

Step1. Let A = (a_{ij}) be an n × n payoff matrix. Obtain a new matrix C, whose columns are obtained by subtracting successive columns of matrix A from its proceeding columns.

Step2. Obtain a new matrix R, whose rows are obtained by subtracting successive rows of matrix A from its proceeding rows.

Step3. Determine the oddments corresponding to ith row of A is defined as the determinant $|C|$, where C_i is obtained from C by deleting its ith row.

Similarly, determine oddments $|R_j|$, where R_j is obtained from R by deleting its jth column.

Step4. Determine the magnitude of oddments corresponding to each row and column of A and write the oddments against their respective rows and columns.

Step5. If the sum of row oddments is equal to sum of column oddments, obtain optimum strategies by the oddments expressed as fractions of the grand total otherwise method fails.

Step6. Calculate the expected value of game corresponding to the optimum strategy.

6. Application of ranking of fuzzy numbers to Game theory:

1. Consider the following fuzzy game problem with payoff as pentagonal fuzzy numbers

B

$$A \quad \begin{bmatrix} (7,8,9,10,11) & (2,0,2,4,6) & (-2,0,2,3,4) \\ (0,1,2,3,4,) & (1,3,5,7,9) & (-4,0,2,4,8) \\ (4,6,8,10,12) & (4,5,6,7,8) & (3,6,9,12,15) \end{bmatrix}$$

By definition of normal pentagonal fuzzy number $\tilde{A}$ is calculated as defined in section 3. Convert the fuzzy game problem into crisp problem, then solved by maximin-minimax principle and find the value of the game.

$\tilde{a}_{11} = (7,8,9,10,11)$	$M_0^p(\tilde{a}_{11}) = 28$
$\tilde{a}_{12} = (2,0,2,4,6)$	$M_0^p(\tilde{a}_{12}) = 10$
$\tilde{a}_{13} = (-2,0,2,3,4)$	$M_0^p(\tilde{a}_{13}) = 5$
$\tilde{a}_{21} = (0,1,2,3,4)$	$M_0^p(\tilde{a}_{21}) = 7$
$\tilde{a}_{22} = (1,3,5,7,9)$	$M_0^p(\tilde{a}_{22}) = 17$
$\tilde{a}_{23} = (-4,0,2,4,8)$	$M_0^p(\tilde{a}_{23}) = 8$
$\tilde{a}_{31} = (4,6,8,10,12)$	$M_0^p(\tilde{a}_{31}) = 26$
$\tilde{a}_{32} = (4,5,6,7,8)$	$M_0^p(\tilde{a}_{32}) = 19$
$\tilde{a}_{33} = (3,6,9,12,15)$	$M_0^p(\tilde{a}_{33}) = 30$

The given fuzzy game problem is reduced in the following payoff matrix

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 28 & 10 & 5 \\ 7 & 17 & 8 \\ 26 & 19 & 30 \end{bmatrix} \end{array}$$

Minimum of 1st row = 5

Minimum of 2nd row = 7

Minimum of 3rd row = 19

Maximum of 1st column = 28

Maximum of 2nd column = 19

Maximum of 3rd column = 30

Max (min) = 19 and Min (max) =19

Here Max (min) = Min (max)

It has equilibrium point and the value of Game = 19

2. Consider the following fuzzy game problem with payoffs as :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} (-0.1,0,0.3,0.5,0.7) & (-3,0,3,7,10) & (-1,0,1,2,3) \\ (0,2,3,4,5) & (0,2,4,6,8) & (-2,0,,2,3,4) \\ (0.8,0.7,0.5,0.3,0.2) & (7,9,12,15,18) & (-4,7,9,10,13) \end{bmatrix} \end{array}$$

By definition of normal pentagonal fuzzy number $\tilde{A}$ is calculated as defined in section 3. Convert the fuzzy game problem into crisp problem, then solve by oddment method.

Step1. We obtain the values of $M_0^p(\tilde{a}_{ij})$ of the given fuzzy game problem and convert the fuzzy game into crisp value problem which is given in the following table:

$\tilde{a}_{11} = (-0.1,0,0.3,0.5,0.7)$	$M_0^p(\tilde{a}_{11})$ = 0.34
$\tilde{a}_{12} = (-3,0,3,7,10)$	$M_0^p(\tilde{a}_{12})$ = 6.4
$\tilde{a}_{13} = (-1,0,1,2,3)$	$M_0^p(\tilde{a}_{13})$ = 4
$\tilde{a}_{21} = (0,2,3,4,5)$	$M_0^p(\tilde{a}_{21})$ = 9.5
$\tilde{a}_{22} = (0,2,4,6,8)$	$M_0^p(\tilde{a}_{22})$ = 14
$\tilde{a}_{23} = (-2,0,2,3,4)$	$M_0^p(\tilde{a}_{23})$ = 5
$\tilde{}a_{31} = (0.8,0.7,0.5,0.3,0.2)$	$M_0^p(\tilde{a}_{31})$ = 1.3
$\tilde{a}_{32} = (7,9,12,15,18)$	$M_0^p(\tilde{a}_{32})$ = 31.6
$\tilde{a}_{32} = (-4,7,9,10,13)$	$M_0^p(\tilde{a}_{33})$ = 13.1

Step2. The given fuzzy game problem is reduced in the following payoff matrix

B

$$A \begin{bmatrix} 3.5 & 6.5 & 2.5 \\ 2.2 & 0.5 & 0.5 \\ 11.5 & 5.5 & 7.5 \end{bmatrix}$$

Minimum of 1st row = 2.5

Minimum of 2nd row = 0.5

Minimum of 3rd row = 5.5

Maximum of 1st column = 11.5

Maximum of 2nd column = 6.5

Maximum of 3rd column = 7.5

Max (min) = 5.5 and Min (max) =6.5

Here Max (min) $\neq$ Min (max)

It has no saddle point.

Step3. To solve the reduced crisp value problem, we apply dominance method. Clearly second column is dominated by first column as all the elements of second column are greater than first column. Hence eliminating second column, we get

B

$$A \begin{bmatrix} 0.34 & 4 \\ 9.5 & 5 \\ 1.3 & 13.1 \end{bmatrix}$$

Again, first row is dominated by the second row as all the elements of first row are less than second row. Hence eliminating first row, we get

B

$$A \begin{bmatrix} 9.5 & 5 \\ 1.3 & 13.1 \end{bmatrix}$$

Now we obtain 2×2 payoff matrix, Since the reduced matrix do not have any saddle point, so we apply oddment method. Thus the augmented payoff matrix is

		Row oddments	
	9.5	5	11.8
	1.3	13.1	4.5
Column oddments	8.1	8.2	

Since the sum of row oddments and column oddments equal to 16.3, the optimum strategies are:

Row player ($0.723, 0.276$) and column player ($0.4969, 0.5030$)

The value of the game is 7.2268.

7. Conclusion:

The paper describes ranking method for pentagonal fuzzy numbers to solve fuzzy game problems. This new ranking method is based on α -cut and plays an important role in real life situations. By using this method fuzzy numbers are converted into crisp numbers so that game problem can be solved by traditional method.

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