Vector Valued Generalized Partial B-Metric Space and Some Common Fixed Point Theorems Endowed with a Graph

Seema Mehra[1], Anuradha[2]

Assistant Prof[1], Research Scholar[2],

Department of Mathematics, M.D.U, Rohtak(HR)

Abstract:

In this paper, first we define vector valued generalized partial b-metric space, then we obtain some common fixed point results for two self- mappings on vector valued generalized partial b-metric space endowed with a graph. These results substantially extend, generalize and improve some well known result in the existence literature.

Keywords: Vector valued generalized partialb-metric space, common fixed point, self-mapping, connected graph.

MSC:47H10, 54H25

1. Introduction

The area of the fixed point theory has wide number of application in applied mathematics and sciences. Recently, the common fixed points of mappings satisfying certain contractive conditions has been studied extensively by many authors.French mathematician Maurice Frechet in 1906 introduced the concept of metric space. After the work of French, several generalization of metric space came out by several mathematicians and one of the generalization is partial metric space that one introduced by Mathews[13] to obtain appropriate mathematical models in the theory of computation.

Bakhtin [1] in 1989, introduced*b*-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to *b*-metric spaces. Since then, various number of articles came out in the improvement of fixed point theory in *b*-metric spaces.

The classical Banach contraction principle was extended for contraction mappings to spaces endowed with vector-valued metrics by Perov in 1964 [2]. Filip, et al. [7] studied the fixed-point property of self-mappings in a generalized metric space (X, d) and generalized the results obtained by Perov.

In recent investigations, the study of fixed point theory endowed with a graph play an important role in many areas. Jachymski[10] was the first who gave two results by giving sufficient condition for f to be a picard operator, if (X,d) endowed with a graph and applied it to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space *C*[0,1]. According to Jachymaski, use the language of graph theory instead of a partial ordering is more convenient.In our paper, we reformulate some important common fixed point results from generalized metric spaces to vector valued generalized partial *b*-metricspaces endowed with a graph, motivated by the theory given in some recent work on metric spaces with a graph (see [3-6, 8,14].The results are extensions of some theorems given by Jabbari et al.(2013) and Kamran et al.(2016).

2.Some Basic Concepts and Notations

Let us recall first some important preliminary concepts and results.

Let Xbe a nonempty set. A mapping $d:X\times X\rightarrow R^m$ is called a vector-valued metric on X if the following properties are satisfied:

1.$d(x,y)\geq O$ for all $x,y\in X$, and $d(x,y)=O$iff $x=y$ where $O=(0,0,0,0,0\ldots\ldots.)$

2. $d(x,y)=d(y,x)$ for all $x,y\in X$;

3. $d(x,y)\leq s[d(x,z)+d(z,y)]$ for all $x,y\in X$.

A nonempty set X endowed with a vector-valued metric dis called avector-valued metric space and it will be denoted by (X,d). If α, $\beta \in R^m$, $\alpha =(\alpha_1, \alpha_2, \ldots, \alpha_m)$,$\beta =(\beta_1, \beta_2, \ldots, \beta_m)$, and $c \in R$, by $\alpha \leq \beta$(resp., $\alpha < \beta$)we mean that $\alpha_i\leq \beta_i$(resp., $\alpha i< \beta i$) for $i\in \{1, 2, \ldots, m\}$ and by$\alpha \leq c$ we mean that $\alpha_i\leq c$ for$i\in \{1, 2, \ldots, m\}$.

Throughout this paper we denote the non-empty set of all m ×m matrices with non-negative elements by $M_{m,m}(R_+)$, the zero m× m matrix by $\overline{0}$ and the identity m ×m matrix by I, and note that $A^0 = I$. A matrix A is said to be convergent to zero if and only if $A^n\rightarrow \overline{0}$ as $n \rightarrow \infty$

Theorem 2.1 ([7]). Let $A \in M_{m,m}(R_+)$, The followings are equivalent.

(i) A is convergent towards zero;

(ii) $A^n\rightarrow \overline{0}$ as $n \rightarrow \infty$;

(iii) the eigenvalues of A are in the open unit disc, that is $|\lambda|< 1$, for every $\lambda \in C$

with$\det(A - \lambda I) = 0$;

(iv) the matrix I -A is nonsingular and

$(I - A)^{-1} = I + A +\ldots\ldots.+ A^n + \ldots\ldots$; (1.1)

(v) The matrix I -A is nonsingular and $(I - A)^{-1}$has nonnegative elements;

(v)for $A^n q \rightarrow \overline{0}$ and $qA^n\rightarrow \overline{0}$ as $n \rightarrow \infty$, for each $q \in R^m$

Remark: Some examples of matrix convergent to zero are

(a) any matrix $A :=\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ where $a, b\in R^+$ and $a+b<1$;

(b) any matrix $A :=\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ where $a, b \in R^+$ and $a+b<1$;

(c) any matrix $A :=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $a, b,c\in R^+$ and $\max\{a,c\}<1$;

For other examples and considerations on matrices which converge to zero, see [8] and [12].

Theorem 2.2 ([2]). Let (X, d) be a complete vector valued generalized metric space and the mapping $f : X \rightarrow X$ with the property that there exists a matrix $A \in M_{m,m}(R_+)$ such that

$d(f(x), f(y)) \leq Ad(x, y)$ for all $x, y \in X$. If A is a matrix convergent towards zero, then

(1) $Fix(f) = \{x*\}$;

(2) the sequence of successive approximations $\{x_n\}$such that $x_n = f^n(x_0)$ is convergent and it has the limit $x*$, for all $x_0\in X$.

(3) One has the following estimation:

$d(x_n, x*) \leq A(I-A)^{-1} d(x_0, x_1)$

We next review some basic notions in graph theory.

Let (X,d) be a vector valued generalized*b*-metric space. We assume that *G* is a reflexive digraph where the set V (G) = Xand the set E(G) of its edges contains no parallel edges. So we can identify *G* with the pair (V(G), E(G)). *G* may beconsidered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1}, we denote the graph obtained from *G* by reversing the direction of edges. Let $\tilde{G}$denote the undirected graph obtained from *G* by ignoring the direction of edges so,$E(\tilde{G}) = E(G) \cup E(G^{-1})$. Actually, it will be more convenient for us to treat $\tilde{G}$as a digraph for which the set of its edges is symmetric.

Our graph theory terminology and notations are standard and can be found in all graph theory books. If x, yare vertices of the digraph *G*, then a path in *G* from *x* to *y*of length *n* (n∈N) is a sequence $(x_i)_{i=0}^{i=n}$of *n* + 1 vertices such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E(\tilde{G})$for *i*= 0, 1,2, … *n*. A graph *G* is connected if there is a path between any two vertices of *G*. *G* is weakly connected if$\tilde{G}$ is connected.On other hand Jachymski [4], generalized the Banach contraction principle on a complete metric space endowed with a graph. He introduced the notion of Banach G-contraction as follows:

Banach G - contraction or Simple G- contraction[10] : A mapping f: X → X is a Banach G-contraction if f preserves edges of G i.e.

$\forall x, y \in X \ ((x,y) \in E(G) \Rightarrow (fx, fy) \in E(G))$

and$f$ decreases weights of edges of G in the following way:

$\exists\, \alpha \in (0,1) \forall\, x, y \in X \ ((x, y) \in E(G) \Rightarrow d(fx, fy) \leq \alpha\, d(x,y))$.

G-continuous[10] : A mapping f: X → X is called G- continuous if given x ∈ X and a sequence $(x_n)_{n \in N}$, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for n ∈ N imply $fx_n \to fx$.

Remark. If f is a G-contraction thenf is both a G^{-1}- contraction and a $\tilde{G}$- contraction.

Let T and S be self- mappings of a set X. Recall that, if y = Tx= Sxfor some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S. The mappings T, S are weakly compatible[17], if for every x ∈ X, the following holds:

T(Sx) = S(Tx) whenever Sx= Tx.

Proposition 2.4[12]. Let S and T be weakly compatible self-maps of a nonempty set X. If S and T have a unique point of coincidence y = Sx= Tx, then y is the unique common fixed point of S and T.

3.Main result

Common fixed point theorem: Let f and g be two self-mappings on complete vector valued generalized partialb- metric space with coefficient s≥ 1endowed with a graph G = (V, E), and G is a reflexive digraph such that V(G)= X and G has no parallel edges. Let mappings f, g :X → X be such that f(X) ⊆g(X). If $x_0 \in X$ is arbitrary, then there exists an element $x_1 \in X$ such that

$fx_0 = gx_1$, since f(X) ⊆ g(X). Proceeding in this way, we can construct a sequence (gx_n) such that

$gx_n = fx_{n-1}$,n, m= 1,2, 3, ….. .By C_{gf}we denote the set of all elements x_0 of X such that

$(fx_n, fx_m) \in E(\tilde{G})$form, n = 0,1,2, …… . In this section, we study the existence of a common fixed point for these mappings. For this we are needed Proposition define in section 2.

Definition 3.1. Let Xbe a nonempty set and $s \geq 1$ be real number. A mappingV_{pb}: $X \times X \rightarrow R^m$ is called a vector-valued generalized partial b- metric on X if the following properties are satisfied:

(1.)$V_{pb}(x, y) \geq V_{pb}(x, x) \geq O$ for all x, y ∈ X,

(2). $V_{pb}(x, x) = V_{pb}(x, y) = V_{pb}(y, y) = O$ iff x = y

(3). $V_{pb}(x, y) = V_{pb}(y, x)$ for all x, y ∈ X;

(4). $V_{pb}(x, y) \leq s[V_{pb}(x, z) + V_{pb}(z, y)] - V_{pb}(z,z)$for all x, y ∈ X.

where O=(0,0,0,0,0……..)i.e zero vector in R^m.

A nonempty set X endowed with a vector-valued generalized partial b- metric $V_{pb}$is called a vector-valued generalized partial b- metric space with coefficient s≥ 1, and it will be denoted by (X, V_{pb}).

Remark: In above definition 3.1,

(4).$V_{pb}(x, y) \leq s[V_{pb}(x, z) + V_{pb}(z, y)] - V_{pb}(z,z)$ for all x, y ∈ X implies that

$V_{pb}(x, y) \leq s[V_{pb}(x, z) + V_{pb}(z, y)]$ for all x, y ∈ X. (5.) and

vector-valued generalized metric space ⇒ vector-valued generalized b- metric space ⇒ vector-valued generalized partial b- metric space.

Example 3.1: Let X = {1, 2, 3, 4} and V_{pb}: $X \times X \rightarrow R^3$ be defined by

$$V_{pb}(x, y) = \begin{cases} |X - Y|^2 + \max\{X, Y\} & if\ x \neq y \\ X & if\ x = y \neq 1 \\ O & if\ x = y = 1 \end{cases}$$

where $X = [1\ \ x\ \ 0]^T$ and $Y = [1\ \ y\ \ 0]^T$ are vectors in R^3.

Then (X, V_{pb}) is a completevector-valued generalized partial b-metric space with coefficient s = 4 > 1.

Solution: Clearly,$V_{pb}(x, y) \geq V_{pb}(x, x) \geq O$ for all x, y ∈ X since vectors X ≥ Y when x ≥ y and $V_{pb}(x, y) = |X - Y|^2 + \max\{X, Y\} \geq V_{pb}(x, x) = X$ when $x \neq y$.

Here, $V_{pb}(x, x) = V_{pb}(x, y) = V_{pb}(y, y) = Oiffx = y = 1$ and it is easy to see

$V_{pb}(x, y) = V_{pb}(y, x)$ for all x, y ∈ X.

Now, we see (4).

It is enough to take the following cases:

1. x = y = z
2. x< y<z
3. x< y = z
4. x> y = z
5. x = y < z

Here we consider case 4.. x > y = z implies X> Y = Z

V_{pb} (x, y) $=|X-Y|^2+\max\{X,Y\}= |X-Y|^2+X$

and

s[V_{pb} (x, z) + V_{pb} (z, y)] -V_{pb} (z,z) = s [$|X-Y|^2+X$] - Z if y = z$\neq$ 1

Clearly, we observe that

V_{pb} (x, y)$\leq$s[V_{pb} (x, z) + V_{pb} (z, y)] -V_{pb} (z,z)

Similarly, we can prove that inequality (4.) hold for all cases.

Definition 3.2: Let (X, V_{pb}) be a vector-valued generalizedpartial b-metric space with coefficient s$\geq$ 1. Let $\{x_n\}$ be any sequence in X and x ∈ X. Then

(i) The sequence $\{x_n\}$ is called convergent sequence with respect to metric V_{pb} , if

$\lim_{n\to\infty} V_{pb}(x_n, x) = V_{pb}(x, x)$.

(ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,V_{pb}) if $\lim_{n\to\infty} V_{pb}(x_n, x_m)$ exists and is finite.

(iii) Space (X,V_{pb}) is said to be a complete generalized vector valued partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X converge and there exists x ∈ X such that

$\lim_{n\to\infty} V_{pb}(x_n, x_m) = \lim_{n\to\infty} V_{pb}(x_n, x) = V_{pb}(x, x)$.

Lemma 3.1: Let(X, V_{pb}) be a generalized vector valued partial b-metric space with coefficient

s$\geq$ 1. Then

- If V_{pb} (x, y) = 0 then x=y
- If $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} V_{pb}(x_n, x_{n+1}) = 0$ then we have

$\lim_{n\to\infty} V_{pb}(x_n, x_n) = 0$ and $\lim_{n\to\infty} V_{pb}(x_{n+1}, x_{n+1}) = 0$

- Limit of the sequence in (X, V_{pb}) is always unique.

Proof: The proof is obvious.

Theorem 3.1: Let (X,V_{pb}) be complete generalized vector valuedpartial b-metric space with with coefficient s and graphG= (V,E) and the mappingsf,g: X $\rightarrow$ X satisfying

$V_{pb}(gx, gy) \leq A\, V_{pb}(fx, fy)$ $\forall\, (fx, fy) \in E(\tilde{G})$. (3.1.1)

where A ∈$M_{m,m}(R^+)$ be a non-zero matrix such that sA convergence to zero and non-singular.Suppose g(X)$\subseteq$f(X) with gof = fogand fis a a G-continuous function with the following property:(∗) for any sequence <x_n>which satisfy $fx_n= gx_{n-1} \Longrightarrow (gx_n, gx_{n-1}) \in E(\tilde{G})$and(∗∗) If x, y are points of coincidence of f and g in X then (x, y) ∈ E($\tilde{G}$) .

Then f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ is arbitrary, then ∃ an element $x_1 \in X$ such that $fx_1 = gx_0$, since g(X) $\subset f(X)$.

Proceeding in this way, we can construct the sequence $(x_n)_{N\cup\{0\}}$, as follows

$fx_n = gx_{n-1}$ (3.1.2)

from (3.1.1) and by $(*)$, we have

$V_{pb}(gx_n, gx_{n-1}) \leq AV_{pb}(fx_n, fx_{n-1})$

$= AV_{pb}(gx_{n-1},\ gx_{n-2})$

$\leq A^2V_{pb}(fx_{n-1},\ fx_{n-2})$

$\leq A^nV_{pb}(fx_1, fx_2)$

$= A^n R$ where $V_{pb}(fx_1, fx_2) = R$

Now, for m > n and using (5.), we have

$V_{pb}(gx_n, gx_m) \leq s[V_{pb}(gx_n, gx_{n+1}) + V_{pb}(gx_{n+1}, gx_m)]$

$\leq [sV_{pb}(gx_n, gx_{n+1}) + s^2V_{pb}(gx_{n+1}, gx_{n+2}) + \ldots + s^{m-n}V_{pb}(gx_{m-1}, gx_m)]$

$\leq [s A^{n+1} R + s^2 A^{n+2} R + s^3 A^{n+3} R + \ldots + s^{m-n} A^m R]$

$\leq s A^{n+1} R [1 + s A + (sA)^2 + \ldots + (sA)^{m-n-1}]$

$\leq s A^{n+1} R(I - (sA)^{m-n}) (I - sA)^{-1}$ where I is identity matrix and $I - s A$ is non-singular.

$$\rightarrow 0 \ as \ n \rightarrow \infty$$

which shows that (gx_n) is a Cauchy sequence in X.Therefore, (fx_n) is also a Cauchy sequence in X by the above computation. By completeness of X, there exist $t \epsilon X$ such that $fx_n \rightarrow t$.

Since $\lim_{n\to\infty} V_{pb}(fx_n, fx_m) = 0 = \lim_{n\to\infty} V_{pb}(fx_n, t) = V_{pb}(t, t)$. Also by definition of g, $gx_n \rightarrow t$ (using 3.1.1). G- continuity of f implies that g is also aG- continous map.

Thus,

$g(fx_n) \rightarrow g(t)$

and

$f(gx_n) \rightarrow f(t)$

Since gof = fog

So,

$g(fx_n) \rightarrow f(t)$

By uniqueness of limit, f(t) = g(t) = u (let).

Therefore, u is point of coincidence of f and g. Next, we show that point of coincidence is unique. Assume that there is another point of coincidence $u*$ in X such that $fx = gx = u*$, for some $x \in X$. By property, we have $(u, u*) \in E(\tilde{G})$.

Then, $V_{pb}(u,\ u*) = V_{pb}(gt, gx) \leq A\ V_{pb}(ft, fx)$

$= A\ V_{pb}(u,\ u*)$

$(I - A)\ V_{pb}(u,\ u*) \leq 0$

$\Rightarrow V_{pb}(u,\ u*) = 0$ i.e $u = u*$

Therefore, f and g have a unique point of coincidence.

Since fog = gof

andf(t) = g(t)

$\Rightarrow g(f(t)) = g(g(t)) = f(g(t))$

i.e $g(g(t)) = f(g(t)) = z$ (let), for some $z \in X$

$\Rightarrow z$ is another point of coincidence in X but we have shown that point of coincidence is unique.
Thus, z= u i.e $g(g(t)) = f(g(t)) = g(t)$

This shows that g(t) is common fixed point of the mappings f and g. Finally, we show that g(t) is unique common fixed point. Let y be another common fixed point of f and g. Therefore, we have, fy = gy = y i.e y be point of coincidencebut we have shown that point of coincidence is unique implies that u = g(t) = y. Thus f and g have unique common fixed point.

Corollary 3.1: Let (X, V_{pb}) be complete generalized vector valued partial b-metric space with graph G = (V, E) and the mappings f,g: $X \to X$ satisfying

$$V_{pb}(g^k(x), g^k(y)) \leq A\, V_{pb}(fx, fy) \quad \forall (fx, fy) \in E(\tilde{G}). \qquad (a)$$

where $A \in M_{m,m}(R^+)$ be a non-zero matrix convergence to zero and $k \in N$ with $s\beta \neq 1$, β is the eigen value of matrix A. . Suppose $g(X) \subseteq f(X)$ with gof = fogand fis a G- continuous function with the following property:
(∗) for any sequence $<x_n>$ which satisfy $fx_n = gx_{n-1} \Longrightarrow (g^k x_n, g^k x_{n-1}) \in E(\tilde{G})$, ∀ k and
(∗∗) If x, y are points of coincidence of f and g in X then $(x, y) \in E(\tilde{G})$.
Thenf and g have a unique common fixed point.
Proof: Let $T = g^k$. For every k >1,

$g^k o f = g^{k-1} o g o f = g^{k-1} o f o g = \ldots\ldots\ldots\ldots\ldots = f o g^k$

i.e T o f = f o T

so, contractive condition(a) becomes

$V_{pb}(T(x), T(y)) \leq A\, V_{pb}(fx, fy)$

Also, $g^k(X) = T(X) \subseteq g(X) \subseteq f(X)$. Therefore, by theorem 3.1, $g^k = T$ and f have a unique common fixed point. Let $u \in X$ be a unique common fixed point of g^k and f. Thus $u = f(u) = g^k(u)$. Since f and g are commutating mappings, then

$g(u) = g(f(u)) = g(g^k(u)) = g^k(g(u)) = f(g(u))$.

Thus g(u) is a common fixed point of g^k and f. Since, the common fixed point of g^k and f was unique. Hence we should have u = f(u) = g(u).

Theorem 3.2. Let (X, V_{pb}) be a vector valued generalizedpartial b-metric space endowed with a graph G and themappings f, g : X → X satisfy

$$V_{pb}(fx,fy) \leq k\, V_{pb}(fx, gx) + l\, V_{pb}(fy,gy) + p\, V_{pb}(fx, gy) + q\, V_{pb}(fy, gx) \qquad (3.2.1)$$

for all $x, y \in X$ with $(fx,fy) \in E(\tilde{G})$, where k, l, p, q are positive numbers such that

$l + k + 2ps < \frac{1}{s}$, $p + q \neq 1$.

Suppose $f(X) \subseteq g(X)$ and f(X) is a complete subspace of X with thefollowing property:

(∗)If (fx_n) is a sequence in X such that $fx_n \to x$ and $(fx_n, fx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exist a subsequence (fx_{n_i}) of (fx_n) such that $(fx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Then f and g have a point of coincidence in X if $C_{g f} \neq \phi$. Moreover, f and g have aunique point of coincidence in X if the graph G has the property :

(∗∗) If x, y are points of coincidence of f and g in X then $(x, y) \in E(\tilde{G})$.

Furthermore,if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Suppose that$C_{g f} \neq \phi$. We can choose an $x_0 \in C_{g f}$ and keep it fixed. Since $f(X) \subseteq g(X)$, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, n = 1,2 ,3, …and $(fx_n, fx_m) \in E(\tilde{G})$, for m, n = 0,1,2,3……., also $(gx_n, gx_m) \in E(\tilde{G})$, for m, n = 1,2,3…….

We shall show that (fx_n) is a Cauchy sequence in f(X).

For proving (fx_n) is a Cauchy sequence, enough to show that (gx_n) is a Cauchy sequence.

For any natural number *n*, we have by using condition (3.2.1) and (4.);

$V_{pb}(gx_{n+1}, gx_n)=V_{pb}(fx_n, fx_{n-1})$

$\leq kV_{pb}(fx_n, gx_n)+lV_{pb}(fx_{n-1}, gx_{n-1})+ pV_{pb}(fx_n\, gx_{n-1})+ qV_{pb}(fx_{n-1}gx_n)$

$\leq kV_{pb}(gx_{n+1}, gx_n) +lV_{pb}(gx_n, gx_{n-1}) + p V_{pb}(gx_{n+1}, gx_{n-1}) + q V_{pb}(gx_n,gx_n)$

$\leq kV_{pb}(gx_{n+1}, gx_n) +lV_{pb}(gx_n, gx_{n-1}) + psV_{pb}(gx_{n+1},gx_n)+ psV_{pb}(gx_n, gx_{n-1})+ q V_{pb} (gx_n,gx_n)$

$\leq (k+ps) V_{pb}(gx_{n+1}, gx_n) +(l +ps+q) V_{pb}(gx_n, gx_{n-1})$

which gives that

$V_{pb}(gx_{n+1}, gx_n) \leq \alpha V_{pb}(gx_n, gx_{n-1})$ (3.2.2)

where$\alpha =\frac{l+ps+q}{1-k-ps} \in (0,1)$, $1-k-ps \neq 0$. By repeated use of condition (3.2.2), we get

$d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_2), \forall n \in N$. (3.2.3)

For m, n ∈ N, m≤n and using condition (3.2.1) and (3.2.3), we have

$V_{pb}(gx_m, gx_n) =V_{pb}(fx_{m-1}, fx_{n-1})$

$\leq kV_{pb}(fx_{m-1}, gx_{m-1}) +lV_{pb}(fx_{n-1}, gx_{n-1}) + p V_{pb}(fx_{m-1}, gx_{n-1}) + q V_{pb}(fx_{n-1}, gx_{m-1})$

$\leq kV_{pb}(gx_m, gx_{m-1}) +lV_{pb}(gx_n, gx_{n-1}) + p V_{pb}(gx_m, gx_{n-1}) + q V_{pb}(gx_n, gx_{m-1})$

Now, by triangle inequality (5.) and (3.2.3)

$pV_{pb}(gx_m, gx_{n-1}) \leq p\{s V_{pb}(gx_m, gx_{m+1}) + s^2V_{pb}(gx_{m+1}, gx_{m+2})+ s^3 V_{pb}(gx_{m+2}, gx_{m+3})+\ldots+ s^{n-m-1}V_{pb}(gx_{n-2}, gx_{n-1})\}$

$\leq p\{ s\, \alpha^m V_{pb}(gx_1, gx_2)+s^2\alpha^{m+1}V_{pb}(gx_1, gx_2) +s^3\alpha^{m+2}V_{pb}(gx_1, gx_2)+\ldots+s^{n-m-1}\alpha^{n-2}V_{pb}(gx_1, gx_2)\} \leq (s\, \alpha^m +s^2\alpha^{m+1} +s^3\alpha^{m+2} +\ldots\ldots +s^{n-m-1}\alpha^{n-2})V_{pb}(gx_1, gx_2)$

$=\alpha^m s\,(1 + s\,\alpha +(s\alpha)^2 +\ldots\ldots +(s\alpha)^{n-m-2})\ p\ V_{pb}(gx_1, gx_2)$

$= ps\, \alpha^m (\frac{1-(s\alpha)^{n-m-1}}{1-s\alpha})\ V_{pb}(gx_1, gx_2)$

andq $(gx_n, gx_{m-1}) \leq q\{ s V_{pb}(gx_{m-1},gx_m) + s^2V_{pb}(gx_m, gx_{m+1}) + s^3V_{pb}(gx_{m+1}, gx_{m+2})+\ldots+ s^{n-m}V_{pb}(gx_{n-2}, gx_{n-1})+ s^{n-m+1}V_{pb}(gx_{n-1},gx_n)\}$

$\leq q\, s\{\alpha^{m-1} + s\, \alpha^m+ s^2 \alpha^{m+1}+\ldots..+ s^{n-m-1}\alpha^{n-2}+ s^{n-m}\alpha^{n-1}\}V_{pb}(gx_1, gx_0)$

$\leq qs\alpha^{m-1}$ {1 +s α + $(s\alpha)^2$ +….+$(s\alpha)^{n-m}$ }$V_{pb}(gx_1, gx_2)$

$\leq qs\,\alpha^{m-1}(\frac{1-(s\alpha)^{n-m+1}}{1-s\alpha})$ $V_{pb}(gx_1, gx_2)$

Therefore,

$V_{pb}(gx_m, gx_n) \leq [\, k\,\alpha^{m-1} + l\,\alpha^{n-1} + ps\,\alpha^m(\frac{1-(s\alpha)^{n-m-1}}{1-s\alpha}) + qs\,\alpha^{m-1}(\frac{1-(s\alpha)^{n-m+1}}{1-s\alpha})]\, V_{pb}(gx_1, gx_2)$

$$\to 0 \qquad as\ m, n \to \infty$$

Therefore, (gx_n) is a Cauchy sequence, so (fx_n) is also a Cauchy sequence in f(X). As f(X) is complete, there existsa $z\in$f(X) such that $fx_n \to u$= fvfor some v$\in X$ and $V_{pb}(u, u)=0$ by using concept $V_{pb}(gx_m, gx_n) \to 0$ $as\ m, n \to \infty$. As $x_0 \in C_{g\,f}$, it followsthat$(fx_n, fx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property (∗), there exists a

subsequence(fx_{n_i}) of (fx_n) such that $(fx_{n_i}, fv) \in E(\tilde{G})$ for all i≥ 1.

Now using conditions (3.2.1) and (3.2.3), we find

$V_{pb}(fv,gv) \leq s[V_{pb}(fv, fx_{n_i}) + V_{pb}(fx_{n_i}, gv)]$ (By triangle inequality (5.))

$\leq s\, V_{pb}(fv, fx_{n_i}) + s[k\, V_{pb}(fx_{n_i}, gx_{n_i}) + lV_{pb}(fv, gv) + p\, V_{pb}(fx_{n_i}, gv) + qV_{pb}(fv, gx_{n_i})]$

$\leq s\, V_{pb}(fv, fx_{n_i}) + s\,[\, kV_{pb}(fx_{n_i}, fx_{n_i-1}) + lV_{pb}(fv, gv) + p\, V_{pb}(fx_{n_i}, gv) + q\, V_{pb}(fv, fx_{n_{i-1}})\,]$

$\Rightarrow (1- sl\,)V_{pb}(fv,gv) \leq s\, V_{pb}(fv, fx_{n_i}) + skV_{pb}(fx_{n_i}, fx_{n_i-1}) + psV_{pb}(fx_{n_i}, gv) +$

$qsV_{pb}(fv, fx_{n_{i-1}})$ (3.2.4)

As$fx_{n_i} \to fv \quad as\ i \to \infty$

i.e$V_{pb}(fv, fx_{n_i}) \to 0 as\ i \to \infty$

implies that

$V_{pb}(gv, fx_{n_i}) \to V_{pb}(gv, fv) as\ i \to \infty$

Therefore, taking $i \to \infty$ in (3.2.4), we get

$(1-sl)V_{pb}(fv,gv) \leq psV_{pb}(fv,gv)$

i.e $(1- sl- sp)\, V_{pb}(fv,gv) \leq 0$

i.e$V_{pb}(fv,gv) = 0$

$\Rightarrow$fv = gv =u

Therefore, u is a point of coincidence of f and g.

Finally, to prove the uniqueness of the point of coincidence, suppose that there is another point of coincidence u^*in X such that fz= gz= $u^*$for some z X.

By property (∗∗), we have $(u, u^*) \in E(\tilde{G})$. Then, by (3.2.1)

$d(u, u^*) = d(fv, fz) \leq kV_{pb}(fv,gv) + l\, V_{pb}(fz, gz) + p\, V_{pb}(fv, gz) + qV_{pb}(fz, gv)$

$\leq$kV_{pb}(u, u) +$l$$V_{pb}$($u^*$, u^*) +p V_{pb}(u, u^*) + qV_{pb}(u^*, u)

Implies that

(1-p-q)V_{pb}(u, u^*) ≤ 0

$\Rightarrow$ u=u^*i.e we have shown that f and g have unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition2.4,f and g have a unique common fixed point in X.

Corollary 3.2: Let (X; V_{pb}) be a vector valued generalized partialb-metric space endowed with a graph G and the mappings f, g : X $\rightarrow$ X satisfy

V_{pb}(fx, fy) $\leq$ k V_{pb}(fx, gx) + l V_{pb}(fy,gy)

for all x, y $\in$X with (fx, fy) $\in$ $E(\tilde{G})$, where k, l are positive numbers such that k+l<$\frac{1}{s}$.

Suppose f(X) $\subseteq$g(X) and f(X) is a complete subspace of X with the property($*$).

Then f and g have a point of coincidence in X if Moreover, f and g have aunique point of coincidence in X if the graph G has the property ($**$).

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof: Put p = q = 0 in theorem 3.2 then we obtain our required result.

Corollary 3.3: Let (X,V_{pb}) be a vector valued generalized b-metric space endowed with a graph G and the mappings f, g :X $\rightarrow$ X satisfy

d(fx, fy) $\leq$ p V_{pb}(fx, gy) + q V_{pb}(fy, gx)

for all x, y $\in$X with (fx, fy) $\in$ $E(\tilde{G})$, where p, q are positive numbers such that sp<$\frac{1}{1+s}$

ors$l$<$\frac{1}{1+s}$.

Suppose f(X) $\subseteq$g(X) and f(X) is a complete subspace of X with the property($*$).

Then f and g have a point of coincidence in X if Moreover, f and g have aunique point of coincidence in X if the graph G has the property ($**$).

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof: Put k =l = 0 in theorem 3.2 then we obtain our required result.

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