Hesitancy Fuzzy Labeling Graph

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Abstract:

The objective of this paper is to introduce the concept of hesitancy fuzzy labeling. A graph is said to be a hesitancy fuzzy labeling graph if it has hesitancy fuzzy labeling. And some properties of hesitancy fuzzy labeling graph and hesitancy fuzzy labeling tree are discussed.

Keywords: Hesitancy Fuzzy Graph, Hesitancy Fuzzy Labeling Graph, Hesitancy Fuzzy Labeling Tree.

AMS Subject Classification (2010): 05C72, 05C78, 03E72.

1. Introduction

A graph labeling is an assignment of numbers to the vertices or edges or both, subject to certain conditions. J.A.Gallian [8] presented a survey on recent results, conjectures and open problems in labeling graph. Labeled graphs serve us useful models for broad range of applications such as coding theory, X-ray, radar, astronomy, circuit design and communication networks, etc.

The concept of fuzzy sets and fuzzy relations were introduced by L.A. Zadeh in 1965 [27]. A. Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975 [23]. The concept of fuzzy labeling and fuzzy magic labeling graph were introduced by A. Nagoor Gani et.al. [11, 12, 13]. S. N. Mishra and Anita Pal discussed Magic labeling of interval-valued fuzzy graph in 2016 [10].

Intuitionistic fuzzy sets [6] and Intuitionistic fuzzy graph [7] were introduced by Krassimir T. Atanassov in 1986 and 1999 respectively. R. Parvathi et.al. propounded the intuitionistic fuzzy graph and its properties [14, 15]. Seema Mehra and Manjeet Singh introduced intuitionistic fuzzy magic labeling graph in 2017 [24].

M. Akram et.al. deliberated interval valued fuzzy graph [2] in 2011 and bipolar fuzzy graph in 2011 [1] and intuitionistic fuzzy hyper graph [3] in 2013. M. Akram and Arooj Adeel introduced m-polar fuzzy labeling graphs with application in 2017 [4, 5]. P.K. Kishore Kumar et.al discussed magic labeling on interval-valued intuitionistic fuzzy graphs in 2017 [9].

Hesitant fuzzy sets introduced by V. Torra in 2010 [25]. T. Pathinathan et.al. introduced hesitancy fuzzy graph in 2015 [16] and discussed various properties in 2016 [17, 18, 19]. N. Vinothkumar and G. Geetharamani discussed operations in hesitancy fuzzy graphs in 2018 [26]. R. Rajeswari et.al. discussed hesitant fuzzy trees in 2018 [20]. M.A. Rifayathali et.al. introduced hesitancy fuzzy graph coloring [22] and hesitancy fuzzy magic labeling graph in 2018 [21].

The concept of labeling for hesitancy fuzzy graph which has a wider applications in fields like network, optimization, medical diagnostic system and remote sensing. So here in this paper some properties of hesitancy fuzzy labeling graph and hesitancy fuzzy labeling tree are introduced and discussed.

2. Preliminaries

2.1. Definition (L.A. Zadeh [27])

Let X be a non-empty set. Then a fuzzy set A in X (i.e., a fuzzy subset A of X) is characterized by a function of the form $\mu_A: X \to [0,1]$, such a function μ_A is called the membership function and for each $x \in X$, $\mu_A(x)$ is the degree of membership of x (membership grade of x) in the fuzzy set A.

In otherwords, $A = \{(x, \mu_A(x))/\, x \in X\}$ where $\mu_A: X \to [0,1]$.

2.2. Definition (A. Rosenfeld [23])

A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0, 1]$ and $\mu: V \times V \to [0, 1]$, where for all $u\,, v\ \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$.

2.3. Definition (A. Nagoorgani et.al. [11])

A graph $G = (\sigma, \mu)$ is said to be a fuzzy labeling graph if $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ is bijective such that the membership value of edges and vertices are distinct and $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

2.4. Definition (A. Nagoorgani et.al. [13])

A fuzzy labeling graph is said to be a fuzzy magic graph if $\sigma(u) + \mu(u, v) + \sigma(v)$ has a same magic value for all $u, v \in V$.

2.5. Definition (Krassimir T. Atanassov [6])

An Intuitionistic Fuzzy set A in a set X is defined as an object of the form

$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/\, x \in X\}$ where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X\,;\, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

2.6. Definition (Krassimir T. Atanassov [7])

Intuitionistic Fuzzy Graph (IFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_1: V \to [0,1]$ and $\nu_1: V \to [0,1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_i) + \nu_1(v_i) \leq 1$, for every $v_i \in V\,, (i = 1,2, \ldots, n)$.

(ii) $E \subset V \times V$ where $\mu_2: V \times V \to [0,1]$ and $\nu_2: V \times V \to [0,1]$ are such that

$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$$
$$\nu_2(v_i, v_j) \leq \max[\nu_1(v_i), \nu_1(v_j)]$$

and $0 \leq \mu_2(v_i, v_j) + \nu_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$.

2.7. Definition (T. Pathinathan et.al [16])

Hesitancy Fuzzy Graph (HFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1: V \to [0,1]$, $\gamma_1: V \to [0,1]$ and $\beta_1: V \to [0,1]$ denote the degrees of membership, non-membership and hesitancy of the element $v_i \in V$ respectively and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$, for every $v_i \in V$, $(i = 1,2, \dots, n)$, where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$, and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$.

(ii) $E \subseteq V \times V$ where $\mu_2: V \times V \to [0,1]$, $\gamma_2: V \times V \to [0,1]$ and $\beta_2: V \times V \to [0,1]$ are such that,

$$\mu_2(v_i, v_j) \le \min[\mu_1(v_i), \mu_1(v_j)]$$
$$\gamma_2(v_i, v_j) \le \max[\gamma_1(v_i), \gamma_1(v_j)]$$
$$\beta_2(v_i, v_j) \le \min[\beta_1(v_i), \beta_1(v_j)]$$

and $0 \le \mu_2(v_i, v_j) + \nu_2(v_i, v_j) + \beta_2(v_i, v_j) \le 1$ for every $(v_i, v_j) \in E$.

Here $(v_i, \mu_{1i,}\gamma_{1i}, \beta_{1i})$ denote the degree of membership, degree of non-membership and degree of hesitancy of the vertex v_i and $(e_{ij,}\mu_{2ij,}\gamma_{2ij}, \beta_{2ij})$ denote the degree of membership, degree of non-membership and degree of hesitancy of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

2.8. Definition (M.A. Rifayathali et.al [22])

The arc (u, v) in hesitancy fuzzy graph G is said to be a strong arc if $\frac{1}{2}\min\{\mu_1(u), \mu_1(v)\} \le \mu_2(u, v)$, $\frac{1}{2}\max\{\gamma_1(u), \gamma_1(v)\} \le \gamma_2(u, v)$ and $\frac{1}{2}\min\{\beta_1(u), \beta_1(v)\} \le \beta_2(u, v)$. Otherwise it is called weakest arc.

2.9. Definition (M.A. Rifayathali et.al [21])

A hesitancy fuzzy labeling graph is said to be a hesitancy fuzzy magic graph if the degree of membership value $(\mu_1(u) + \mu_2(u, v) + \mu_1(v))$ remain equal for all $u, v \in V$, degree of non-membership value $(\gamma_1(u) + \gamma_2(u, v) + \gamma_1(v))$ remain equal for all $u, v \in V$ and degree of hesitancy value $(\beta_1(u) + \beta_2(u, v) + \beta_1(v))$ remain equal for all $u, v \in V$. We denote a hesitancy fuzzy magic graph by $M_0(G^*) = (m_\mu(G^*), m_\gamma(G^*), m_\beta(G^*))$ where the magic membership value denoted by $m_\mu(G^*)$, the magic non-membership value denoted by $m_\gamma(G^*)$ and the magic hesitancy value denoted by $m_\beta(G^*)$.

2.10. Definition (R. Rajeswari [20])

If $v_i, v_j \in V \subseteq G$, the μ-strength of connectedness between two vertices v_i and v_j is

$$\mu^\infty(v_i, v_j) = sup\{\mu^k(v_i, v_j) / \; k = 1, 2, \dots, n \; and \; i \ne j\},$$

The γ-strength of connectedness between two vertices v_i and v_j is

$$\gamma^\infty(v_i, v_j) = inf\{\gamma^k(v_i, v_j) / \; k = 1, 2, \dots, n \; and \; i \ne j\},$$

The β-strength of connectedness between two vertices v_i and v_j is

$\beta^{\infty}(v_i, v_j) = sup\{\beta^k(v_i, v_j)/ k = 1,2,\ldots,n \text{ and } i \neq j\}$,

The μ-strength, γ -strength and β- strength of connectedness between two vertices v_i and v_j in G are denoted by $\mu_G^{\infty}(v_i, v_j)$, $\gamma_G^{\infty}(v_i, v_j)$ and $\beta_G^{\infty}(v_i, v_j)$ respectively.

2.11. Proposition (R. Rajeswari [20])

Let G be a hesitant fuzzy graph such that G is a cycle. Then a vertex is a hesitant fuzzy cut vertex of G if and only if it is a common vertex of two hesitant fuzzy bridges.

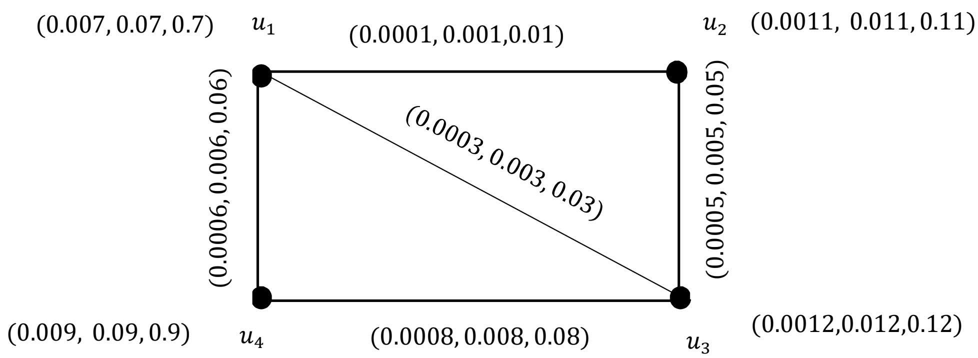
3. Hesitancy Fuzzy Labeling Graph

3.1. Definition

A graph $G = (V, E)$ is said to be hesitancy fuzzy labeling graph if $\mu_1: V \to [0,1]$, $\gamma_1: V \to [0,1]$, $\beta_1: V \to [0,1]$, $\mu_2: V \times V \to [0,1]$, $\gamma_2: V \times V \to [0,1]$ and $\beta_2: V \times V \to [0,1]$ are bijective such that the degrees of vertices and edges are distinct in membership, non-membership and hesitancy.

3.2. Example

An example for a hesitancy fuzzy labeling graph is given below



HFLG G

Figure 3.1

3.3. Definition

A cycle graph G^* is said to be a hesitancy fuzzy labeling cycle graph if it has hesitancy fuzzy labeling.

3.4. Definition

The hesitancy fuzzy labeling graph $H = (V', E')$ is called a hesitancy fuzzy labeling sub graph of $G = (V, E)$ if $\mu_1'(v_i) \leq \mu_1(v_i)$, $\gamma_1'(v_i) \leq \gamma_1(v_i)$, $\beta_1'(v_i) \leq \beta_1(v_i)$ for all $v_i \in V$, $(i = 1,2, \ldots, n)$ and $\mu_2'(v_i, v_j) \leq \mu_2(v_i, v_j)$, $\gamma_2'(v_i, v_j) \leq \gamma_2(v_i, v_j)$, $\beta_2'(v_i, v_j) \leq \beta_2(v_i, v_j)$ for all $(v_i, v_j) \in E$.

3.5. Theorem

If H is hesitancy fuzzy labeling subgraph of G then the strength of connectedness of (v_i, v_j) in H is less than the strength of connectedness of (v_i, v_j) in G for all $(v_i, v_j) \in E$.

Proof.

Let G be any hesitancy fuzzy labeling graph and H be its subgraph. Let (v_i, v_j) be any path in G. Since H is a sub graph, $\mu_1'(v_i) \leq \mu_1(v_i)$, $\gamma_1'(v_i) \leq \gamma_1(v_i)$, $\beta_1'(v_i) \leq \beta_1(v_i)$ and $\mu_2'(v_i, v_j) \leq \mu_2(v_i, v_j)$, $\gamma_2'(v_i, v_j) \leq \gamma_2(v_i, v_j)$, $\beta_2'(v_i, v_j) \leq \beta_2(v_i, v_j)$. Which implies that the strength of connectedness of (v_i, v_j) in H is less than the strength of connectedness of (v_i, v_j) in G for all $(v_i, v_j) \in E$.

3.6. Theorem

Union of any two hesitancy fuzzy labeling graphs G_1 and G_2 is also a hesitancy fuzzy labeling graph, if the degrees of edges between G_1 and G_2 are distinct in membership, non-membership and hesitancy.

Proof.

Let G_1 and G_2 be any two hesitancy fuzzy labeling graphs with distinct degrees of edges between G_1 and G_2. Then $\mu_{11}, \gamma_{11}, \beta_{11}$, $\mu_{12}, \gamma_{12}, \beta_{12}$, $\mu_{21}, \gamma_{21}, \beta_{21}$ and $\mu_{22}, \gamma_{22}, \beta_{22}$ be the hesitancy fuzzy sets of V_1, V_2, X_1 and X_2 respectively. Therefore $\mu_1(x) = \mu_{11}(x)$, $\gamma_1(x) = \gamma_{11}(x)$, $\beta_1(x) = \beta_{11}(x)$ if $x \in V_1 - V_2$ and $\mu_2(x, y) = \mu_{21}(x, y)$, $\gamma_2(x, y) = \gamma_{21}(x, y)$, $\beta_2(x, y) = \beta_{21}(x, y)$ if $x, y \in X_1 - X_2$. Similarly for $\mu_{12}, \gamma_{12}, \beta_{12}$ and $\mu_{22}, \gamma_{22}, \beta_{22}$. And $\mu_1(x) = \max\{\mu_{11}(x), \mu_{12}(x)\}$, $\gamma_1(x) = \max\{\gamma_{11}(x), \gamma_{12}(x)\}$, $\beta_1(x) = \max\{\beta_{11}(x), \beta_{12}(x)\}$ if $x \in V_1 \cap V_2$ and $\mu_2(x, y) = \max\{\mu_{21}(x, y), \mu_{22}(x, y)\}$, $\gamma_2(x, y) = \max\{\gamma_{21}(x, y), \gamma_{22}(x, y)\}$, $\beta_2(x, y) = \max\{\beta_{21}(x, y), \beta_{22}(x, y)\}$ if $x, y \in X_1 \cap X_2$. Hence clearly $\mu_1 = \mu_{11} \cup \mu_{12}$, $\gamma_1 = \gamma_{11} \cup \gamma_{12}$, $\beta_1 = \beta_{11} \cup \beta_{12}$ and $\mu_2 = \mu_{21} \cup \mu_{22}$, $\gamma_2 = \gamma_{21} \cup \gamma_{22}$, $\beta_2 = \beta_{21} \cup \beta_{22}$.

3.7. Theorem

If G^* is a cycle then the hesitancy fuzzy labeling cycle G has exactly only one weakest arc.

Proof.

Since hesitancy fuzzy labeling μ_2, γ_2 and β_2 are bijective, there exists only one arc with minimum weight, say (u, v). It is obvious that the removal of (u, v) from G does not reduce the strength of connectedness, which implies (u, v) is the weakest arc of G. Hence there exists only one weakest arc in any hesitancy fuzzy labeling graph, if G^* is a cycle.

3.8. Corollary

No cycle is a hesitancy fuzzy cycle in hesitancy fuzzy labeling graph.

3.9. Theorem

Let G be a hesitancy fuzzy labeling graph such that G^* is a cycle, then it has $(n - 1)$ bridges.

Proof.

Let G^* be a cycle with hesitancy fuzzy labeling. By theorem 3.7, we will get only one weakest arc. We know that weakest arc is not a hesitancy fuzzy bridge. Which implies the removal of any arc except the weakest arc will reduces the strength of connectedness. Hence every hesitancy fuzzy labeling cycle have $(n-1)$ bridges.

3.10. Theorem

If G^* is a cycle with hesitancy fuzzy labeling then it has $(n-2)$ cut nodes.

Proof.

By theorem 3.9, every hesitancy fuzzy labeling cycle has $(n-1)$ hesitancy fuzzy bridges, (ie) it will have only one weakest arc, say (u, v). Therefore other than u and v, all the remaining $(n-2)$ nodes are common node of two hesitancy fuzzy bridge. Hence by proposition 2.11, hesitancy fuzzy labeling cycle has $(n-2)$ cut nodes.

3.11. Theorem

If G^* is a cycle with hesitancy fuzzy labeling then, the graph has exactly two end nodes.

Proof.

By theorem 3.7, G has exactly only one weakest arc, say (u, v), which implies u and v are end nodes. Hence every hesitancy fuzzy labeling cycle graph has exactly two end nodes.

3.12. Theorem

If G^* is a cycle with hesitancy fuzzy labeling then the node of G is either a cut node or end node.

Proof.

The proof is obvious by theorem 3.10 and 3.11.

3.13. Theorem

If G is a hesitancy fuzzy labeling cycle graph, then every hesitancy fuzzy bridge is strong and vice versa.

Proof.

Let G be a hesitancy fuzzy labeling cycle graph with n nodes. By theorem 3.7, G has exactly only one weakest arc and also by theorem 3.9, G has $(n-1)$ hesitancy fuzzy bridges. Now we claim that all these $(n-1)$ bridges are strong. Let us choose an edge (u_i, u_{i+1}) from $(n-1)$ edges. Since G is a cycle there exists two paths between the nodes u_i and u_{i+1}. (ie) one path with $\mu_2(u_i, u_{i+1}) > 0$, $\gamma_2(u_i, u_{i+1}) > 0$, $\beta_2(u_i, u_{i+1}) > 0$ and the other path with $\mu_2(u_i, u_{i+n}, \dots, u_{i+1}) > 0$, $\gamma_2(u_i, u_{i+n}, \dots, u_{i+1}) > 0$, $\beta_2(u_i, u_{i+n}, \dots, u_{i+1}) > 0$. Therefore $\mu_2^\infty(u_i, u_{i+1}) = \mu_2(u_i, u_{i+1})$, $\gamma_2^\infty(u_i, u_{i+1}) = \gamma_2(u_i, u_{i+1})$ and $\beta_2^\infty(u_i, u_{i+1}) = \beta_2(u_i, u_{i+1})$. Which implies that (u_i, u_{i+1}) is a strong arc. By repeating this process for the remaining edges we will get $(n-1)$ strong arcs and converse is obvious.

3.14. Theorem

If G is a hesitancy fuzzy labeling graph, then G has at least one hesitancy fuzzy bridge.

Proof.

Let G be a hesitancy fuzzy labeling graph. Choose an edge (u, v) such that $\mu_2(u, v)$, $\gamma_2(u, v)$, $\beta_2(u, v)$ are the maximum in the set of all values of $\mu_2(u_i, v_i)$, $\gamma_2(u_i, v_i)$, $\beta_2(u_i, v_i)$ respectively for all $u_i, v_i \in V$. Therefore $\mu_2(u, v) > 0$, $\gamma_2(u, v) > 0$, $\beta_2(u, v) > 0$ and there exist some edge (x, y) distinct from (u, v) such that $\mu_2(x, y) < \mu_2(u, v)$, $\gamma_2(x, y) < \gamma_2(u, v)$ and $\beta_2(x, y) < \beta_2(u, v)$. Now we claim that (u, v) is a hesitancy fuzzy bridge. If we remove (u, v) from G, then in its sub graph H. We have the strength of connectedness between the vertices u and v in H is less than the strength of connectedness between the vertices u and v in G. Hence by theorem 3.13, (u, v) is a hesitancy fuzzy bridge.

3.15. Remark

The converse of the above theorem is not true. Proof is obvious.

3.16. Theorem

If G is a connected hesitancy fuzzy labeling graph then there exists a strong path between any pair of vertices.

Proof.

Let G be a connected hesitancy fuzzy labeling graph and let (u, v) be any pair of vertices. Which implies $\mu_2^\infty(u, v) > 0$, $\gamma_2^\infty(u, v) > 0$ and $\beta_2^\infty(u, v) > 0$. Now choose any edge (u, w) in (u, v), if $\mu_2(u, w) = \mu_2^\infty(u, w)$, $\gamma_2(u, w) = \gamma_2^\infty(u, w)$ and $\beta_2(u, w) = \beta_2^\infty(u, w)$ then it is strong. Otherwise choose some other edge, say (u, x) which satisfies $\mu_2(u, x) = \mu_2^\infty(u, x)$, $\gamma_2(u, x) = \gamma_2^\infty(u, x)$ and $\beta_2(u, x) = \beta_2^\infty(u, x)$. By repeating this process we can find a path in (u, v) in which all arcs are strong.

3.17. Theorem

Every hesitancy fuzzy labeling graph has at least one weakest arc.

Proof.

Let G be a hesitancy fuzzy labeling graph and let (u, v) be an edge of G such that $\mu_2(u, v)$, $\gamma_2(u, v)$ and $\beta_2(u, v)$ are the minimum of all other μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s respectively. If we remove (u, v) from G it does not reduces the strength of any path. In other words, after the removal, in its sub graph H, we have the strength of connectedness between the vertices u and v in G is less than the strength of connectedness between the vertices u and v in H. Which implies (u, v) is neither a hesitancy fuzzy bridge nor a strong arc. Therefore it must be one of the weakest arcs.

3.18. Theorem

For any hesitancy fuzzy labeling graph G, $\delta(G)$ is a hesitancy fuzzy end node of G such that the number of arcs incident on $\delta(G)$ is at least two.

Proof.

Let G be a hesitancy fuzzy labeling graph and there exist at least one node v with degree $\delta(G)$. Which implies the arcs which are incident on v may have lower membership value and it is not possible to have all the arcs which are incident on v as the weakest arc, as μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective. Therefore it must have a strong neighbor. Hence, $\delta(G)$ is an end node of G.

3.19. Theorem

Every hesitancy fuzzy labeling graph has at least one end nodes.

Proof.

It is trivial that for any hesitancy fuzzy labeling graph there exists at least one node with degree $\delta(G)$. Therefore by theorem 3.18, $\delta(G)$ is an end node of G.

3.20. Theorem

Every hesitancy fuzzy labeling graph has at least one cut node.

Proof.

As G is a hesitancy fuzzy labeling graph, there exists at least one weakest arc, say (u, v). Then there exists at least one strongest path $\mu_2^{\infty}(u, v)$, $\gamma_2^{\infty}(u, v)$ and $\beta_2^{\infty}(u, v)$ between the nodes u and v, other than the arc (u, v), say (u, x, v). Therefore x is a cut node of G.

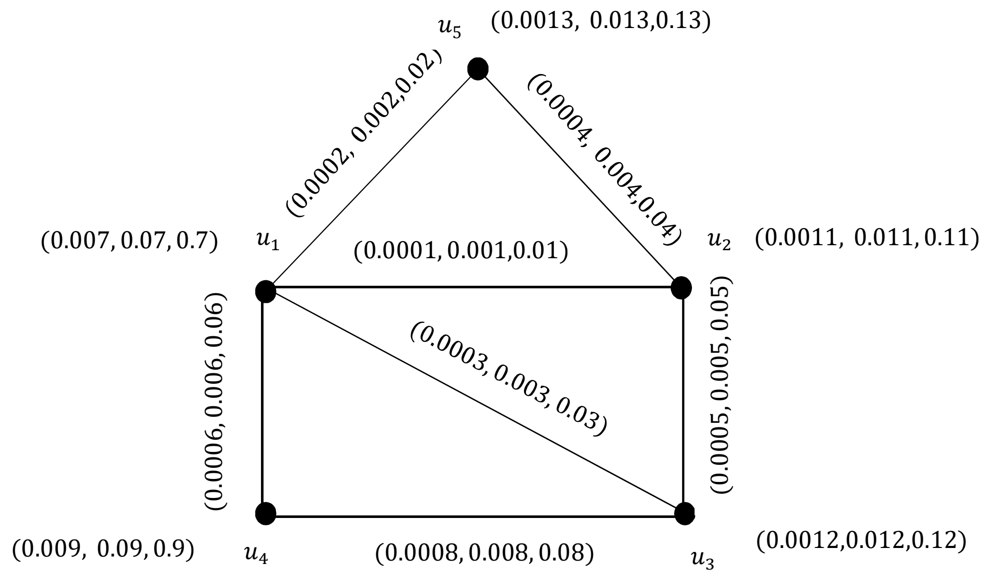
4. Hesitancy Fuzzy Labeling Tree

4.1. Definition

A graph $G = (V, E)$ is said to be a hesitancy fuzzy labeling tree, if it has hesitancy fuzzy labeling and a hesitancy fuzzy spanning sub graph $F = (V, E^*)$ which is a tree, where for all arcs (u, v) not in F, $\mu_2(u, v) < \mu_2^{*\infty}(u, v)$, $\gamma_2(u, v) < \gamma_2^{*\infty}(u, v)$ and $\beta_2(u, v) < \beta_2^{*\infty}(u, v)$.

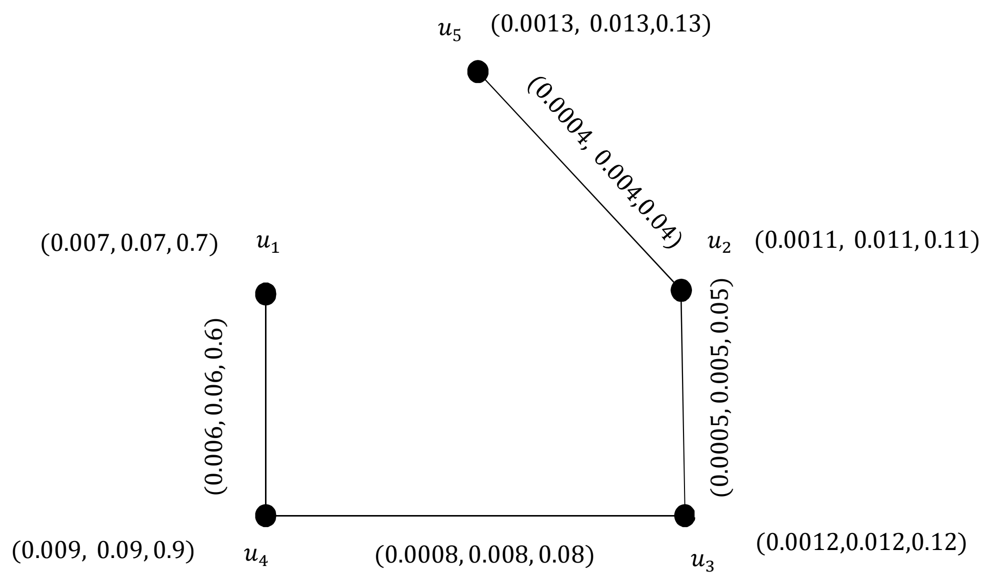
4.2. Example

An example for a hesitancy fuzzy labeling tree and hesitancy fuzzy spanning tree are given below



HFLG G

Figure 4.1



HFLG F

Figure 4.2

4.3. Theorem

If G is a hesitancy fuzzy labeling tree, then the arcs of F are hesitancy fuzzy bridges of G.

Proof.

Let G be a hesitancy fuzzy labeling tree and F be its spanning sub graph. Let (u,v) be an arc in F. Then $\mu_2'^{\infty}(u,v) < \mu_2(u,v) \leq \mu_2^{\infty}(u,v)$, $\gamma_2'^{\infty}(u,v) < \gamma_2(u,v) \leq \gamma_2^{\infty}(u,v)$ and $\beta_2'^{\infty}(u,v) < \beta_2(u,v) \leq \beta_2^{\infty}(u,v)$, which implies the arc (u,v) is a hesitancy fuzzy bridge of G. Conversely, if (u,v) is not a hesitancy fuzzy bridge of G then $\mu_2'^{\infty}(u,v) > \mu_2^{\infty}(u,v) \geq \mu_2(u,v)$, $\gamma_2'^{\infty}(u,v) > \gamma_2^{\infty}(u,v) \geq \gamma_2(u,v)$ and $\beta_2'^{\infty}(u,v) > \beta_2^{\infty}(u,v) \geq \beta_2(u,v)$, which implies (u,v) is not an arc of F.

4.4. Theorem

Every hesitancy fuzzy labeling graph is a hesitancy fuzzy labeling tree.

Proof.

Let G be any hesitancy fuzzy labeling graph. Since μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective, each and every vertex of G will have at least one arc as hesitancy fuzzy bridge. Therefore a spanning sub graph F will exist, such that whose arcs are hesitancy fuzzy bridges. Hence by theorem 4.3, every hesitancy fuzzy labeling graph is a hesitancy fuzzy labeling tree.

4.5. Remark

The above theorem 4.4 is not true for general hesitancy fuzzy graph.

4.6. Theorem

If $G = (V,E)$ is a hesitancy fuzzy labeling tree then its spanning sub graph $F = (V,E^*)$ is also a hesitancy fuzzy labeling graph.

Proof.

Let $G = (V,E)$ be a hesitancy fuzzy labeling tree, by the definition of hesitancy fuzzy labeling $\mu_1: V \to [0,1]$, $\gamma_1: V \to [0,1]$, $\beta_1: V \to [0,1]$, $\mu_2: V \times V \to [0,1]$, $\gamma_2: V \times V \to [0,1]$ and $\beta_2: V \times V \to [0,1]$ are bijective in G. Since F is its hesitancy fuzzy spanning sub graph of G, $\mu_2(u,v) = \mu_2^*(u,v)$, $\gamma_2(u,v) = \gamma_2^*(u,v)$ and $\beta_2(u,v) = \beta_2^*(u,v)$ if $(u,v) \in E^*$, which implies bijection is preserved in F. Hence F is a hesitancy fuzzy labeling graph.

4.7. Remark

All the properties of hesitancy fuzzy labeling graph hold good for hesitancy fuzzy labeling tree. As in the hesitancy fuzzy graph, here also internal nodes of F are cut nodes since the arcs are hesitancy fuzzy bridges. Here also the hesitancy fuzzy spanning sub graph F is unique and which is the maximum spanning tree. And as μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective one cannot conclude that the lower weighted arc will not be there in F.

4.8. Theorem

If G is a hesitancy fuzzy labeling tree and F is its spanning sub graph, then $(G-F)^*$ is a tree.

Proof.

Let G be a hesitancy fuzzy labeling tree, such that G^* is not a tree. By the definition of hesitancy fuzzy labeling tree there exists a spanning sub graph F, which is a tree. By theorem 4.3, the arcs of F are hesitancy fuzzy bridges of G. Therefore $(G-F)^*$ contains no hesitancy fuzzy bridge. By theorem 3.9, hesitancy fuzzy labeling cycles have $(n-1)$ hesitancy fuzzy bridges. Therefore $(G-F)^*$ contains no cycle. Hence $(G-F)^*$ is a tree.

4.9. Remark

The above theorem is not true, if G^* is complete.

4.10. Theorem

Let G be a hesitancy fuzzy labeling tree and F be its spanning sub graph such that G^* is complete. Then $d_G(u) \neq d_F(u)$ for all $u \in V$.

Proof.

Since G^* is complete, it will contain many cycles. By theorem 3.17, every cycle has one arc as weakest arc, which will not be there in F, since the arcs of F are hesitancy fuzzy bridges. Hence $d_G(u) \neq d_F(u)$ for all $u \in V$.

4.11. Remark

The above theorem 4.10 is not true for general hesitancy fuzzy tree and other hesitancy fuzzy labeling trees.

4.12. Theorem

If G is a hesitancy fuzzy labeling tree such that G^* is a cycle then its spanning sub graph F has $(n-1)$ hesitancy fuzzy bridges.

Proof.

It follows from theorem 3.9.

4.13. Remark

The above theorem 4.12 is true for all hesitancy fuzzy labeling trees.

4.14. Theorem

Let G be a hesitancy fuzzy labeling tree such that G^* is complete. Then every hesitancy fuzzy bridge of G is strong and the converse is also true.

Proof.

Let (u,v) be an arc of a hesitancy fuzzy spanning sub graph F, which is a hesitancy fuzzy bridge by theorem 4.3, Therefore by definition $\mu_2(u,v) \geq \mu_2^{*\infty}(u,v) = \mu_2'^{\infty}(u,v)$, $\gamma_2(u,v) \geq \gamma_2^{*\infty}(u,v) = \gamma_2'^{\infty}(u,v)$ and $\beta_2(u,v) \geq \beta_2^{*\infty}(u,v) = \beta_2'^{\infty}(u,v)$. Thus (u,v) is a strong arc of G. Conversely let (u,v) be a strong arc of G, then $\mu_2(u,v) \geq \mu_2'^{\infty}(u,v) = \mu_2^{*\infty}(u,v)$, $\gamma_2(u,v) \geq \gamma_2'^{\infty}(u,v) = \gamma_2^{*\infty}(u,v)$ and $\beta_2(u,v) \geq \beta_2'^{\infty}(u,v) = \beta_2^{*\infty}(u,v)$. Thus (u,v) is an arc of F. Hence (u,v) is a hesitancy fuzzy bridge of G.

4.15. Theorem

Let $G = (V, E)$ be a hesitancy fuzzy labeling tree and $F = (V, E^*)$ be the hesitancy fuzzy spanning sub graph of G, for all (u, v) not in F, then $\mu_2^{*\infty}(u, v), \gamma_2^{*\infty}(u, v)$ and $\beta_2^{*\infty}(u, v)$ are not equal to the height of G.

Proof.

Let us choose an arc (u, v) not in F, which implies $(u, v) \in G$ and which is not a hesitancy fuzzy bridge of G, since the arcs of F are hesitancy fuzzy bridge of G. By the definition of hesitancy fuzzy labeling tree, if (u, v) is not in F, then $\mu_2(u, v) < \mu_2^{*\infty}(u, v)$, $\gamma_2(u, v) < \gamma_2^{*\infty}(u, v)$ and $\beta_2(u, v) < \beta_2^{*\infty}(u, v)$. Since F is a tree, there exist only one path between u and v. Therefore the strength of connectedness between u and v is equal to the weight of the weakest arc. Since μ_{2ij}'s, γ_{2ij}'s, β_{2ij}'s and ${\mu^*}_{2ij}$'s, γ_{2ij}^*'s, β_{2ij}^*'s are bijective, there exists only one weakest arc. This implies $\mu_2^{*\infty}(u, v)$, $\gamma_2^{*\infty}(u, v)$ and $\beta_2^{*\infty}(u, v)$ is not equal to maximum of μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s respectively. Hence $\mu_2^{*\infty}(u, v), \gamma_2^{*\infty}(u, v)$ and $\beta_2^{*\infty}(u, v)$ are not equal to the height of G.

4.16. Theorem

If G is a hesitancy fuzzy labeling tree then there exists a unique strong path between any two nodes of G.

Proof.

If G^* is a tree, then it is done. Now choose a path (u, v) from a hesitancy fuzzy labeling tree G, such that $\mu_2(u_i, v_i) > 0$, $\gamma_2(u_i, v_i) > 0$, $\beta_2(u_i, v_i) > 0$ for all $1 \leq i \leq n$. If the arcs are strong, then $\mu_2(u_i, v_i) \geq \mu_2'^{\infty}(u_i, v_i)$, $\gamma_2(u_i, v_i) \geq \gamma_2'^{\infty}(u_i, v_i)$ and $\beta_2(u_i, v_i) \geq \beta_2'^{\infty}(u_i, v_i)$. Hence each arc of $\mu_2(v_{i-1}, v_i)$, $\gamma_2(v_{i-1}, v_i)$ and $\beta_2(v_{i-1}, v_i)$ are greater than its strength of connectedness respectively. Similarly choose another path between u and v, which is possible since G is connected. But as μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective, getting another strong path is not possible. Hence strong path between any two nodes is unique.

4.17. Definition

A hesitancy fuzzy labeling graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that V_1 and V_2 are hesitancy fuzzy independent sets.

4.18. Definition

A hesitancy fuzzy labeling graph $G = (V, E)$ is connected if there exists a strong path between any pair of vertices.

4.19. Theorem

Every hesitancy fuzzy labeling tree is hesitancy fuzzy bipartite graph.

Proof.

Since G is a hesitancy fuzzy labeling tree, it is connected. By definition 4.18, there exists a strong path between any two nodes of G. Therefore there exists a hesitancy fuzzy independent sets V_1 and V_2, such that the strong arc of the path have one node in V_1 and other in V_2. If G have a strong cycle then bipartite is not possible but strong cycle of any length will not exist in G, since μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective.

4.20. Theorem

If G is a hesitancy fuzzy labeling tree such that G^* is $K^*_{1,n}$, then G is a complete bipartite graph.

Proof.

It is trivial that G is a hesitancy fuzzy labeling tree if G^* is a tree. Therefore $K^*_{1,n}$ is a hesitancy fuzzy labeling tree which is also a complete bipartite graph because $K^*_{1,n}$ graph can be partition into two non-empty hesitancy fuzzy independent sets V_1 and V_2, such that $V_1 = \{v\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. All the arcs of G are strong arc. Therefore the node $v \in V_1$ is a strong neighbor of $v_1, v_2, \ldots, v_n \in V_2$.

4.21. Corollary

(i) Every hesitancy fuzzy labeling graph is not a complete bipartite graph.

(ii) $K^*_{2,n}$ is not a complete bipartite graph.

4.22. Theorem

If G is a hesitancy fuzzy labeling graph with $n \geq 4$, it has at least one node as cut node in each independent set.

Proof.

Since G is a hesitancy fuzzy labeling tree it has a hesitancy fuzzy spanning sub graph F such that F^* is a tree. Therefore all the node of G will exist in F. By theorem 4.19, the nodes of G can be partitioned into two non-empty hesitancy fuzzy independent sets V_1 and V_2. By theorem 4.3, the arcs of F are hesitancy fuzzy bridges which are also strong, since every hesitancy fuzzy bridge is strong. Now choose any path $\{u, v, w, x\}$, then there exist two internal nodes v and w. By proposition 2.11, v and w are hesitancy fuzzy cut nodes. Since v and w are strong neighbor, $v \in V_1$ and $w \in V_2$.

5. Algorithm for finding the spanning subgraph F of a hesitancy fuzzy labeling tree G, such that G^* is complete

5.1. Algorithm

Step 1: Consider a hesitancy fuzzy labeling tree such that G^* is complete with $|V| = n$.

Step 2: Choose a cycle arbitrarily and remove the weakest arc.

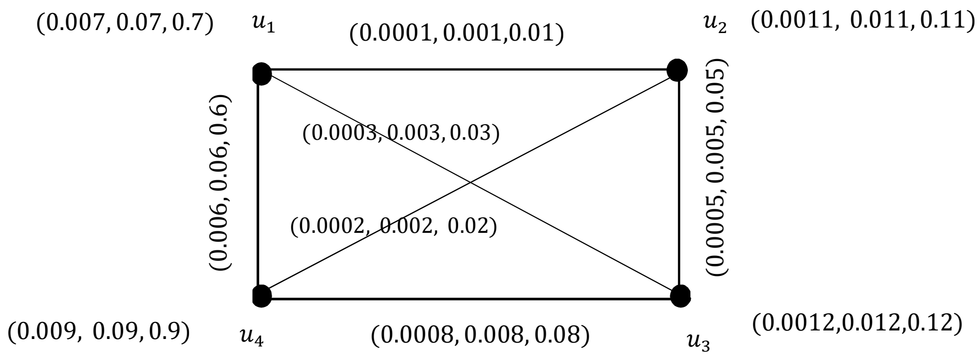
(there exists only one weakest arc, since μ_{2ij}'s, γ_{2ij}'s and β_{2ij}'s are bijective)

Step 3: Repeat Step 2 until no cycle remains.

Step 4: The resulting graph is the spanning subgraph F of a hesitancy fuzzy labeling graph G, whose arcs are hesitancy fuzzy bridges.

5.2. Example

Let us consider a complete hesitancy fuzzy labeling tree G is given below



HFLG G

Figure 5.1

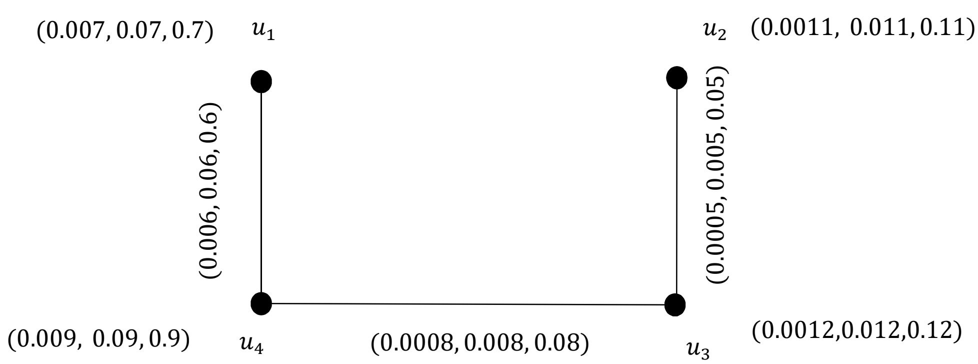
The above algorithm is illustrated with the above hesitancy fuzzy labeling tree.

Step 1 : Figure 5.1 is a hesitancy fuzzy labeling tree with $|V| = 4$ and G^* is complete.

Step 2, 3: Choose a cycle $u_2u_3u_4$ and remove the weakest arc (u_2, u_4).

Similarly choose the remaining cycles and remove the weakest arc.

Step 4 : The following resulting graph in figure 5.2 is the spanning subgraph F of a hesitancy fuzzy labeling graph G, with $(n-1)$ hesitancy fuzzy bridges, which is a tree.



HFLG F

Figure 5.2

6. Conclusion

Hesitancy fuzzy graph have numerous application in the modeling of real life systems where the level of information inherited in the system varies with respect to time and have different level of precision. Most of the actions in real life are time dependent, symbolic models used in expert system are more effective than traditional one. This paper introduces the concept of hesitancy fuzzy labeling graph. It also has discussed results related to hesitancy fuzzy labeling graph and hesitancy fuzzy labeling tree. In future we extend this concept to interval valued hesitancy fuzzy graphs, hesitancy fuzzy hyper graphs and hesitancy fuzzy soft graphs.

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