Numerical Solution of Partial Differential Equations Using Laguerre Wavelets Collocation Method

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Abstract- In this paper, An accurate numerical solution for the partial differential equations (PDE's) with different physical conditions through Laguerre wavelets is presented. In this method we use truncated Laguerre wavelet expansions to convert the PDE's into systems of algebraic equations. Some examples are solved by present method to demonstrate the validity and applicability of present technique. Obtained results are close to the exact solution.

I. INTRODUCTION

Partial differential equations (PDEs) arise in many branches of applied mathematics such as, Hydrodynamics, Fluid dynamics and Mathematical biology etc., In recent years, more importance has been shifted from analytical techniques for PDE's to numerical techniques due to principal of numerical methods is that solutions could be obtained for many problems which are not ready to analytical treatment. Since 1990's [1] wavelet techniques are applying for solving PDEs. The investigation of the numerical solutions for PDEs plays an important role in the study of equations with different physical phenomena. Even though, in latest years, numerical analysis [2] has considerably been developed to be used for partial equations. In the current work, Laguerre wavelets have been applied. In most cases the Laguerre wavelets coefficients have been calculated by collocation method.

Many of powerful methods have been developed by the mathematicians they are as follows, Laguerre wavelets method [3], Haar method [4], B-Spline Wavelet method [5], Chebyshev wavelets method [6], continuous wavelet bases method [7] etc., Our aim of the present work is to solving the PDE's by Laguerre wavelets collocation method which is simple, fast and guarantees the necessary accuracy for a relative small number of grid points.

The arrangements of this article is as follows: In Section 2 we describe properties of Laguerre wavelets and function approximation. Section 3 describe the Laguerre wavelet method. In Section 4 some numerical examples are solved by applying the Laguerre wavelet method of this article. Finally a conclusion is in Section 5.

2. Laguerre wavelets and function approximation

Wavelets represent a family of functions which are generated from dialation and translation of a single function called mother wavelet. Where the dialation parameter a and translation parameter b varies continuously, then family of continuous wavelets is defined as[7]:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi(\frac{x-b}{a}), \forall \ a, b \in R, a \neq 0. \quad (1)$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\psi(a_0^k x - nb_0), \forall a, b \in R, a \neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}(x)$ forms an orthonormal basis. Laguerre wavelets are defined as:

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!} L_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where $m = 0,1, \ldots, M-1$ and $n = 1,2, \ldots, 2^{k-1}$ where k is assumed any positive integer. Here $L_m(x)$ are Laguerre polynomials of degree m with respect to weight function $W(x) = 1$ on the interval $[0, \infty)$ and satisfies the following reccurence formula $L_0(x) = 1$, $L_1(x) = 1 - x$,

$$L_{m+2}(x) = \frac{(2m+3-x)L_{m+1}(x) - (m+1)L_m(x)}{m+2} \quad where\ m = 0,1,2, \ldots$$

Function approximation

Any square integrable function g(x) defined over $[0,1)$ may be expanded by Laguerre wavelets as: $g(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x)$ where $\psi_{n,m}(x)$ is given in Eqn. (2), $C_{n,m} = < g(x), \psi_{n,m}(x) >$ and $<,>$ denotes inner product. We approximate $g(x)$ by truncating the series represented in Eqn. (3) as,

$$g(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = A^T\psi(x)$$

where A and $\psi(x)$ are $2^{k-1}M \times 1$ matrix,

$$A^T = [C_{1,0}, \ldots C_{1,M-1}, C_{2,0}, \ldots C_{2,M-1}, \ldots C_{2^{k-1},0}, \ldots C_{2^{k-1},M-1}]$$

$$\psi(x) = [\psi_{1,0}(x), \ldots \psi_{1,M-1}(x), \psi_{2,0}(x), \ldots \psi_{2,M-1}(x), \ldots \psi_{2^{k-1},0}(x), \ldots \psi_{2^{k-1},M-1}(x)]$$

Similarly, an arbitrary function of two variables $y(x,t)$ defines over $[0,1) \times [0,1)$, can be expressed into Laguerre wavelets basis as: $y(x,t) \approx \psi^T(t)K\psi(x)$

where $\psi^T(t) = (\psi_{1,0}(t), \ldots \psi_{1,M-1}(t), \psi_{2,0}(t), \ldots \psi_{2,M-1}(t), \ldots \psi_{2^{k-1},0}(t), \ldots \psi_{2^{k-1},M-1}(t))$, $\psi(x) = (\psi_{1,0}(x), \ldots \psi_{1,M-1}(x), \psi_{2,0}(x), \ldots \psi_{2,M-1}(x), \ldots \psi_{2^{k-1},0}(x), \ldots \psi_{2^{k-1},M-1}(x))^T$ and $K = [a_{i,j}]_{N\times N}$, $N = 2^{k-1}M$.

3. Method of solution

Laguerre wavelets together with collocation method used to solve PDE's. Consider the general BBM equation as;

$$\alpha y_t(x,t) + \delta y_{xx}(x,t) = \mu(x,t,y) \tag{4}$$

with initial and boundary conditions,

$$y(x,0) = f(x), \quad 0 \le x \le 1$$

and

$$y(0,t) = g_0(t), \quad y(1,t) = g_1(t), \quad \forall t \ge 0$$

where α, δ are real constants and $f(x), g_0(t), g_1(t)$ *and* $\mu(x,t,y)$ are continuous real valued functions. Let us assume that,

$$y_{xxt}(x,t) \approx \psi^T(t) K \psi(x) \tag{5}$$

$$\psi^T(t) = (\psi_{1,0}(t), \dots \psi_{1,M-1}(t), \psi_{2,0}(t), \dots \psi_{2,M-1}(t), \dots \psi_{2^{k-1},0}(t), \dots \psi_{2^{k-1},M-1}(t)) \tag{6}$$

$$\psi(x) = (\psi_{1,0}(x), \dots \psi_{1,M-1}(x), \psi_{2,0}(x), \dots \psi_{2,M-1}(x), \dots \psi_{2^{k-1},0}(x), \dots \psi_{2^{k-1},M-1}(x))^T \tag{7}$$

$$K = [a_{j,j}]_{N\times N}, N = 2^{k-1}M \tag{8}$$

K represents $N \times N$ Laguerre wavelets coefficients to be determined. Now integrate Eqn.(5) with respect to *t* from *0* to *t*.

$$y_{xx}(x,t) = y_{xx}(x,0) + \int_0^t \psi^T(t) \mathbf{K} \psi(x) dt \tag{9}$$

Now integrate Eqn.(9) with respect to *x* from *0* to *x*.

$$y_x(x,t) = y_x(0,t) + y_x(x,0) - y_x(0,0) + \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx \tag{10}$$

Now integrate Eqn.(10) with respect to *x* from *0* to *x*.

$$y(x,t) = y(0,t) + x(y_x(0,t) - y_x(0,0)) + y(x,0) - y(0,0) + \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx \tag{11}$$

put $x = 1$ in Eqn.(11) and by given conditions, we get

$$y_x(0,t) - y_x(0,0) = g_1(t) - g_0(t) + f(0) - f(1) - \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx|_{x=1} \tag{12}$$

Substitute Eqn.(12) in Eqn.(11) and Eqn.(10), we get

$$\begin{aligned} y_x(x,t) = y_x(x,0) + g_1(t) - g_0(t) + f(0) - f(1) - \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx|_{x=1} \\ + \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx \end{aligned} \tag{13}$$

and

$$\begin{aligned} y(x,t) = y(0,t) + x(g_1(t) - g_0(t) + f(0) - f(1) - \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx|_{x=1} \\ + y(x,0) - y(0,0) + \int_0^x \int_0^x \int_0^t \psi^T(t) K \psi(x) dt dx dx \end{aligned} \tag{14}$$

Now differentiate Eqn.(14) with respect to t, we get

$$\begin{aligned} y_t(x,t) = y_t(0,t) + x(g_1'(t) - g_0'(t) - \int_0^x \int_0^x \psi^T(t) K \psi(x) dx dx|_{x=1} \\ + \int_0^x \int_0^x \psi^T(t) K \psi(x) dx dx \end{aligned} \tag{15}$$

Substituting Eqn.(15), Eqn.(14), Eqn.(13) and Eqn.(5) in Eqn.(4) and collocate the obtained equation using following collocation points $x_i, t_i = \frac{2i-1}{M^2}$, $i = 1,2,\dots,M$. Then solve obtained system by suitable solver. We obtain the Laguerre wavelets coefficients $a_{j,j}$, where $j = 1,2,\dots,2^{k-1}M$, then substitute these obtained Laguerre wavelets coefficients in Eqn.(14) will contribute the Laguerre wavelets based numerical solution of Eqn.(4).

4. Numrical experiments

Test Problem 1. Consider the linear PDE [4],

$$y_t(x,t) = y_{xx}(x,t) + \cos(x)$$

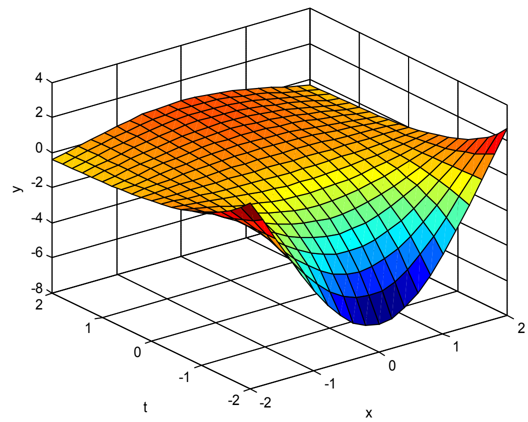
With initial condition

$$y(x,0) = 0 \quad 0 \leq x \leq 1$$

And boundary conditions

$$y(0,t) = 1 - e^{-t}, \quad y(1,t) = \mathrm{Cos}(1)(1 - e^{-t}), \quad \forall t \geq 0$$

The Analytical solution is $\mathrm{Cos}(x)(1 - e^{-t})$. The space-time graph of the numerical solution for $k = 1, M = 9$ is shown in Fig.1. Fig.2 represents the comparison of numerical and exact solution at different values of t.



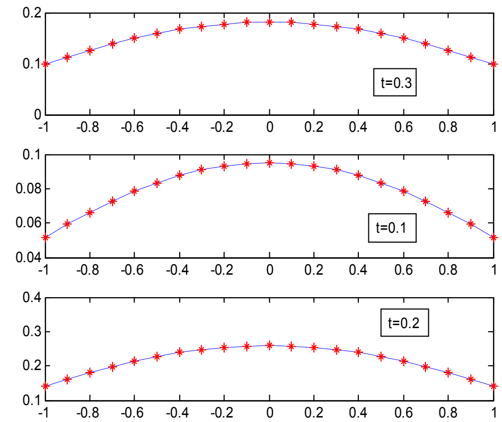


Fig. 1. *Approximate solution of Example 1.* **Fig. 2.** *Comparison of numerical and exact solution at different values of* t *of Example 1.*

Test Problem 2. Consider the nonlinear partial differential equation of the form [8]

$$y_t(x,t) = y_{xx}(x,t) - y(x,t)\big(1 - y(x,t)\big)\big(1 - y(x,t)\big)$$

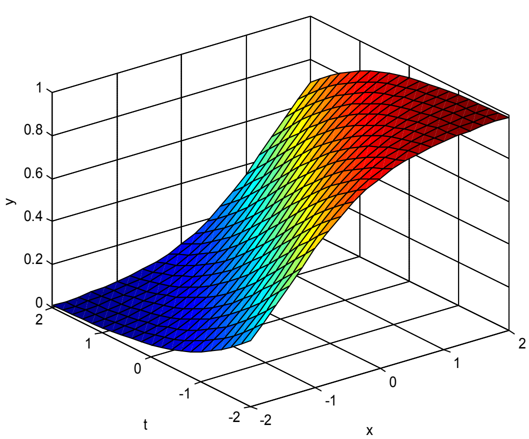
with initial condition

$$y(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right) \quad 0 \leq x \leq 1$$

and boundary conditions

$$y(0,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-t}{4}\right), \quad y(1,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(1 - \frac{t}{\sqrt{2}}\right)\right), \quad \forall t \geq 0$$

The exact solution is $\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{t}{\sqrt{2}}\right)\right)$. The space-time graph of the approximate solution at $k = 1, M = 9$ is shown in Fig.3. Fig.4 represents the comparison of numerical and exact solution at different values of t.



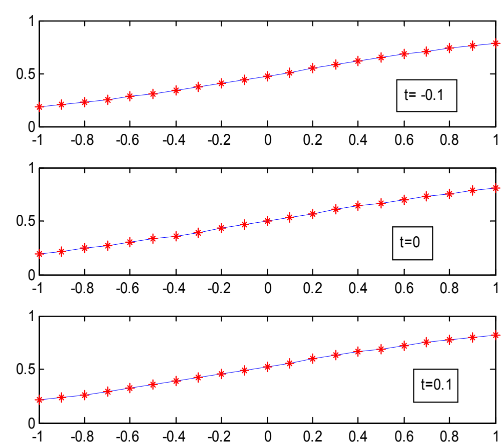


Fig. 3. *Approximate solution of Example 2.* **Fig. 4.** *Comparison of numerical and exact solution at different values of* $\mathbf{t}$ *of* ***Example 2.***

5. Conclusion

In this paper, We have solved linear and nonlinear partial differential equations by Laguerre wavelets based collocation method for different physical conditions. This technique is easy to implement in computer programs and we can extend this scheme for higher order also with slight modification in the present method.

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