Laguerre Wavelet Based Numerical Method for the Solution of Third order Non-linear Delay Differential Equations with damping

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Abstract- In this paper, Laguerre wavelet based numerical method is developed for the solution of third order non-linear Delay Differential Equations. It is based on Laguerre wavelet basis. To confirm the validity and significance of the technique, it is tested with examples. The calculated numerical results are compared with the exact results to exemplify the strength and efficiency of the proposed method.

I. INTRODUCTION

In recent years, researchers from various fields are broadcasting their concentration on delay differential equations for the reason that of their applications in scientific and technical fields, as in biology, economics, viscoelasticity, signal transmission, etc. we have many sources in the pasture of dde's and their applications like halsmith [1], driver[2], halanay [3] kuang [4] and gopalsamy[5] etc.

In the mathematical description of a physical process, one can easily guess that, the performance of the course of action considered depends only on the present state. However, there exist many situations where this assumption is not fulfilled and the use of a classical model in systems analysis and their intend may lead to poor performance. In such conditions, it is better to consider that system's behavior includes also in turn of the previous state, such types of systems are called as Time-delay systems. Many of the processes; both natural and artificial involve time delays. In fact, time delays occur so often, in almost every condition, that to pay no attention to them is to reduce realism.
Conveniently, we have a number of numerical methods for solving DDE's, the significant one's are the Homotopy perturbation method (HPM)[6], variational iteration method[7], Adomian decomposition method (ADM) [8], Bellman's method of steps [9], Legendre Wavelet Method[10], Radau IIA method[11] and Spline Methods[12,13] etc.

The Wavelet theory is one of the relatively new & a budding area in the field of science and Engineering. It has been applied in a large range of engineering disciplines such as Signal & Image processing analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy functioning etc.

Wavelets permit the perfect representation of a fusion of functions and operaters. Wavelets are believed as basis functions $\psi_{i,j}(x)$ in continuous time. Basis is a set of functions which are linearly independent and these linearly independent functions can be used to create all admissible functions say $y(t)$. It is represented in wavelet space as $y(t)=\sum_{i,j} c_{ij}\psi_{ij}(x)$. Extraordinary feature of the wavelet basis is that all functions $\psi_{i,j}(x)$ are constructed from a single mother wavelet $\psi(x)$ which is a small pulse. Typically set of linearly independent functions (basis) produced by translation and dilation of mother wavelet.

In this paper, we have developed Laguerre wavelet Based Numerical Method for the numerical solution of third order non-linear DDEs. In proposed method, unknown function appearing in the delay differential equations is replaced by series expansions of Laguerre basis. After collocating this by appropriate collocation points with given conditions then a system of nonlinear equations are obtained which can be solved using iterative methods to give the unknown coefficients hence the solution is obtained by substituting these unknowns in the assumed approximate solution.

The paper is prepared as follows: In Section 2 presents some basic definitions and properties of Laguerre wavelets. Section 3 is dedicated to description of the proposed method. In section 4, we present some numerical experiments to provide evidence of higher accuracy and efficiency of the proposed method. The conclusion and discussions are presented In section5.

2. BASIC DEFINITIONS AND PROPERTIES OF LAGUERRE WAVELETS

Wavelets represents a family of functions constructed from dialation and translation of a single function $\psi(x)$ called mother wavelet. When the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets as [14],

$$\psi_{a,b}(x)=|a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right), \forall\, a,b\in R, a\neq 0. \tag{1}$$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0.$ We have the following family of discrete wavelets:

$$\psi_{k,n}(x)=|a|^{k/2}\,\psi(a_0^k x-nb_0), \forall a,b\in R, a\neq 0 \tag{2}$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$,then $\psi_{k,n}(x)$ forms an orthonormal basis.

The Laguerre wavelets $\psi_{k,n}(x)=\psi(k,n,m,x)$ involve four arguments n =1, 2, 3..., 2^{k-1}, k is assumed any positive integer, m is the degree of the Laguerre polynomials and it is the normalized time. They are defined on the interval [0, 1) as

$$\psi_{k,n}(x)=\begin{cases} 2^{k/2}\,\overline{L_m}(2^k x-2n+1) & ,\dfrac{n}{2^{k-1}}\leq x\leq \dfrac{n}{2^{k-1}} \\ 0 & ,\text{otherwise}\end{cases} \tag{3}$$

Where $\overline{L_m}(x)=\dfrac{L_m}{m!}$ (4)

m = 0, 1, 2... M-1. Here $\overline{L_m}(x)$ are the Laguerre polynomials of degree m with respect to the weight function W(x) =1 on the interval $[0,\infty]$ and satisfy the following recurrence formula $L_0(x)=1$, $L_1(x)=1-x$ and

$$L_{m+2}(x)=\frac{(2m+3-x)L_{m+1}(x)-(m+1)L_m(x)}{m+2},\ m=0,1,2,3,\ldots.$$

3. LAGUERRE WAVELET BASED NUMERICAL METHOD

Approximation of Function:

A function defined over [0, 1) can be expanded as a Laguerre wavelet series as follows:

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}K_{n,m}\psi_{n,m}(x)$$

where $\psi_{n,m}(x)$ is given in the equation (3). We approximate $y(x)$ by reduced series as,

$$y(x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(x)=K^T\psi(x)$$

here K and $\psi(x)$ are $2^{k-1}M\times 1$ matrices given by,

$$K^T=[K_{1,0},\ldots K_{1,M-1,}K_{2,0},\ldots K_{2,M-1,},\ldots,K_{2^{k-1},0},\ldots K_{2^{k-1},M-1}].$$

$$\psi^T=[\psi_{1,0},\ldots\psi_{1,M-1,}\psi_{2,0},\ldots\psi_{2,M-1,},\ldots,\psi_{2^{k-1},0},\ldots\psi_{2^{k-1},M-1}].$$

Since the truncated wavelets series can be an approximate solution of equations one has an error function $E(x)$ for y(x) as follows:

$E(x)=\left|\mathrm{y(x)}-K^T\psi(\mathrm{x})\right|$.

Proposed Method:

The solution of Delay Differential Equation can be stretched as a Laguerre wavelet series as:

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}K_{n,m}\psi_{n,m}(x)$$

where $\psi_{n,m}(x)$ is given by the equation (3). We estimate y(x) by reduced series

$$y_{k,M}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(x)=K^T\psi(x) \tag{5}$$

Here, $K^T=[K_{1,0},\ldots K_{1,M-1,}K_{2,0},\ldots K_{2,M-1,},\ldots,K_{2^{k-1},0},\ldots K_{2^{k-1},M-1}].$

$$\psi^T=[\psi_{1,0},\ldots\psi_{1,M-1,}\psi_{2,0},\ldots\psi_{2,M-1,},\ldots,\psi_{2^{k-1},0},\ldots\psi_{2^{k-1},M-1}].$$

Then a total number of $2^{k-1}M$ conditions should exist to determine the $2^{k-1}M$ coefficients.

$K_{10},K_{11},\ldots,K_{1M-1},K_{20},K_{21},\ldots,K_{2M-1},\ldots,K_{2^{K-1}0,}K_{2^{K-1}1,}\ldots,K_{2^{K-1}M-1,}$

Case: When given equation is of order three then there are three initial conditions and namely

$$\sum_{k=0}^{m_1-1}{}_{i,k}\,y^k(0)=\lambda_i \quad i=0,1,\ldots m_1-1,\ m_1=2\ where\ K_{i,k}\neq 0. \tag{6}$$

Now, we observe that there should be $2^{k-1}M-2$ extra conditions to recuperate the unknown coefficients $K_{n,m}$. This we are carrying by putting Equation (5) in Equation (1):

$$\frac{\mathrm{d}^{m_1}}{\mathrm{dx}^{m_1}}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(x)=\sum_{i=1}^{J}\sum_{k=0}^{m_1-1}p_{i,k}(x)\frac{d^k}{dx^k}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(\alpha_i x)+f(x) \tag{7}$$

Suppose, equation (7) is exact at $2^{k-1}M-3$ points X_i's as follows:

$$\frac{\mathrm{d}^{m_1}}{\mathrm{dx}^{m_1}}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(x_{i1})=\sum_{i=1}^{J}\sum_{k=0}^{m_1-1}p_{i,k}(x_{i1})\frac{d^k}{dx^k}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}K_{n,m}\psi_{n,m}(\alpha_i x_{i1})+f(x_{i1}) \tag{8}$$

x_{i1}'s are limit points of the following sequence:

$$\{x_{i1}\}=\{\frac{1}{2}(1+\cos\frac{(i_1-1)}{2^{k-1}M-1})\}\quad i_1=2,3,\ldots \tag{9}$$

Combine Equation (8) and (7) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients $K_{n,m}$.

Same procedure is repeated for equations of other and higher order also.
Note: Here k =1 fixed and M should be greater than or equal to the order of given equations.

4. IMPLEMENTATION OF METHOD

In this section, three experiments of non-linear DDE are given and comparisons are made to illustrate the efficiency of the method.

Example 1. Consider the following non linear third order delay differential equation [17].

$$y'''(t) = -1 + 2y^2(t/2) \tag{10}$$

with conditions $y(0) = 0$, $\frac{dy}{dx}(0) = 1$ and $\frac{d^2y}{dx^2}(0) = 0$.

The exact solution of equation (10) is $y(t) = \sin t$ which we have solved y using our technique with k=1 and M=5. We compared these result with exact solution.

Table 1: Comparison of the exact value $y(t)$ and approximation value $y_n(t)$ for k=1 and M=5, of the example 1.

t	Exact value	Approximation Value
0.1	0.099833416646828	0.099804882124144
0.2	0.198669330795061	0.198459599911919
0.3	0.295520206661340	0.294870482052178
0.4	0.389418342308651	0.388005485988397
0.5	0.479425538604203	0.476894197918732
0.6	0.564642473395035	0.560627832796017
0.7	0.644217687237691	0.638359234327763
0.8	0.717356090899523	0.709302874976165
0.9	0.783326909627483	0.772734855958090

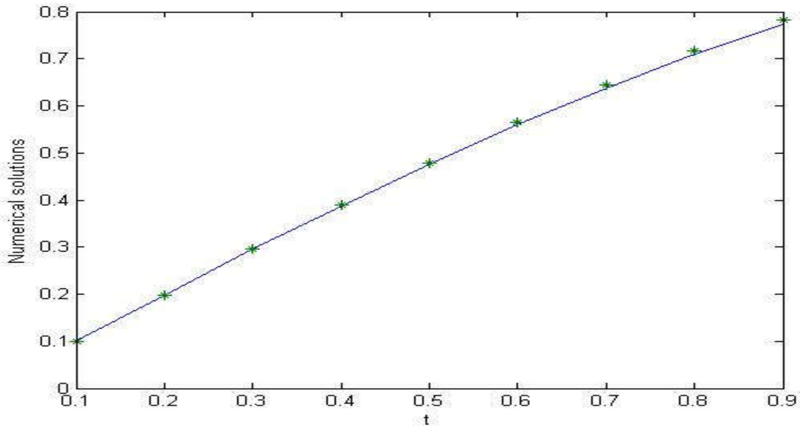

Figure1: Comparison of Laguerre Wavelet Method solutions with Exact Solutions

Example 2. Consider the third order non linear delay differential equation [18].

$$y'''(t)+\frac{1}{4t^2}y'+(1-\frac{1}{4t^2})y(t-\frac{3\pi}{2})=0 \qquad t>t-\frac{3\pi}{2}>\frac{1}{2} \qquad (11)$$

Subjected to the conditions: $y(0)=-1$, $y'(0)=0$ and $y''(0)=1$.

The analytic solution of this problem is $y(t)=-\cos t$ .We have solved this problem by applying the Laguerre Wavelet Method (LWM) with M=5 and k=1.The obtained results are given in Table2.

Table 2. Comparison of the exact value $y(t)$ and approximation value $y_n(t)$ for k=1 and M=5 of the example 2.

t	Exact value	Approximation Value
0	-1.0000	-1.0000
0.05	-0.9988	-0.9987
0.1	-0.9950	-0.9950
0.15	-0.9888	-0.9887
0.2	-0.9801	-0.9798
0.25	-0.9689	-0.9683
0.3	-0.9553	-0.9543
0.35	-0.9394	-0.9377
0.4	-0.9211	-0.9186
0.45	-0.9004	-0.8968
0.5	-0.8776	-0.8724

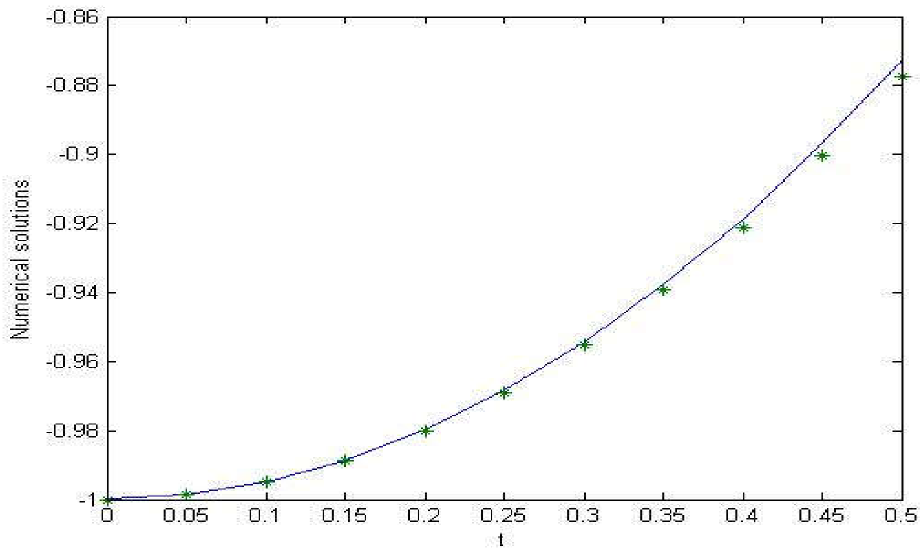

Figure2. Comparison of Laguerre Wavelet Method solutions with Exact Solutions

Example3. Consider the following third order non linear Delay Differential Equation **[18].**

$$y'''(t)+e^{-2t+2}y'(t)+\frac{1}{e}y(t-1)(1+t^2y(t-1))=0 \qquad (12)$$

Where $t>t-1>1$. Subjected to the initial conditions $y(0)=0$, $y'(0)=-1$ and $y''(0)=1$.

The exact solution of this problem is known to be $y(t)=e^{-t}$.

We solve (12) using our technique with k=1 and M=5. The Solution obtained by our method is excellent and holds with the exact solution as shown in the figure.3

Table 3. Comparison of the exact value $y(t)$ and approximation value $y_n(t)$ for k=1 and M=5 of the example 3

t	Exact value	Approximation Value
0	1.0000	1.0000
0.0250	0.9753	0.9753
0.0500	0.9512	0.9512
0.0750	0.9277	0.9278
0.1000	0.9048	0.9050
0.1250	0.8825	0.8829
0.1500	0.8607	0.8613
0.1750	0.8395	0.8405
0.2000	0.8187	0.8202
0.2250	0.7985	0.8006
0.2500	0.7788	0.7817
0.2750	0.7596	0.7634
0.3000	0.7408	0.7457
0.3250	0.7225	0.7287
0.3500	0.7047	0.7123

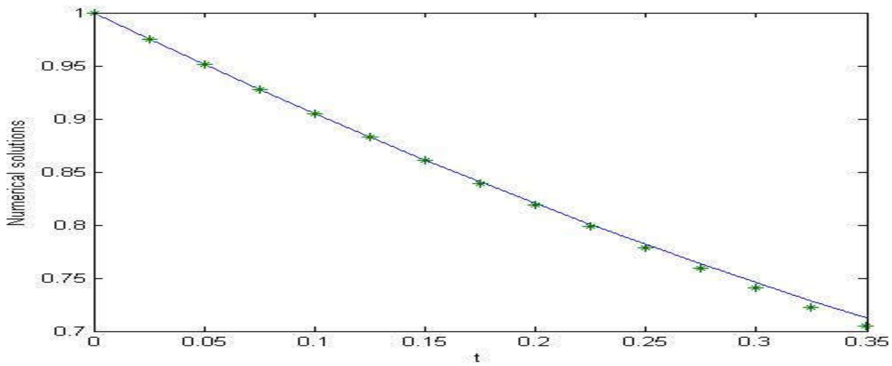

Figure3. Comparison of Laguerre Wavelet Method solutions with Exact Solutions

5. CONCLUSIONS

Laguerre Wavelet Based Numerical Method is developed for the Solution of third order Non-linear Delay Differential Equations with damping. Due to the fast converging property of Laguerre Wavelet polynomials this method of solution is easy to implement and yields desired accuracy within a few terms. As we observed, the method works excellently for third order and same may be extended to other order non linear DDEs which are diversified physical problems of more complexity in nature. This exhibits one of the other good advantages of the method in spite of its simplicity. The advantage of this method over others is that it has less computational complexity; therefore it reduces the runtime and provides the solution at high accuracy. Hence the proposed method is reliable, powerful and promising. We trust that the efficiency of this method gives it a much wider applicability which should be explored further.

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