Haar Wavelet Filters Multigrid Method for the Solution of Non-linear Partial Differential Equation

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Abstract- This paper presents a new method for the solution of the non-linear partial differential equation by the combination of lifting scheme and haar wavelet filters multigrid method. It uses the haar wavelet filters and lifting technique for the efficiency of the multigrid method, which is tested in the Burgers equation. Computational results using a code based on our method is presented and compared with the exact solution to show the efficiency of the method.

KEYWORDS: Burger's equation, Haar wavelet filters; Lifting Technique**;** Newton's method.

I. INTRODUCTION

Numerous researchers have introduced different methods to resolve the Burger's equation as a replica of turmoil. For the solution of Burger's equation dissimilar authors developed the distinct types of methods and they are presented in these references [1-11]. We present a new approach and solved the Burger's equation efficiently by the combination of lifting scheme and haar wavelet filters multigrid method.

2. PRELIMINARIES OF HAAR WAVELET FILTERS

Filters are initiated from wavelets with compact support and are such that, $h_n = 0$ for $n < 0$ and $n > L$, in which L is the length of the filter. The minimum requirements for these compact FIR filters are:

(i)The length of the scaling filter h_n must be even. (ii) $\sum_n h_n = \sqrt{2}$ (iii) $\sum_n (h_n \cdot h_{n-2k}) = \delta(k)$, in which $\delta(k)$ is the Kronecker delta, such that, it is equal to 1 for $k = 0$ or 0 for $k = 1$.

Haar wavelet filter coefficients [9]: Low pass: $h=\left[h_0,\ h_1\right]^T=\left[\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right]^T$ & high pass: $g=\left[g_0,\ g_1\right]^T=\left[\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right]^T$.

3. HAAR WAVELET FILTERS MULTIGRID METHOD

Haar wavelet filters property is explored by Multigrid method for create the steps of matrices, i.e., $A^{j-1}=R_j^{j-1}A^jP_{j-1}^j$, where $R_{nh}^{\frac{nh}{2}}=$ restriction operator $R_j^{j-1}:V_j\to V_{j-1}$, prolongation operator $P_{\frac{nh}{2}}^{nh}=P_{j-1}^j:V_{j-1}\to V_j$, $P_{j-1}^j=(R_j^{j-1})^T$, V_j denote the space of the variables of the system at level j and V_{j-1} one of the sub space of V_j. In this approach the low-pass filter coefficients h_i are used in the building of a matrix R, which is used as a restriction operator. Haar wavelet filters restriction operator $R_{nh}^{\frac{nh}{2}}$ from level j to j-1 takes the following form,

$$R_j^{j-1}=R_{nh}^{\frac{nh}{2}}=\begin{bmatrix} h_1 & h_0 & 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & h_1 & h_0 & 0 & 0 & 0 & . & . & .0 \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ . & & & & & & & & & . \\ 0 & 0 & 0 & . & . & . & . & .0 & h_1 & h_0 \end{bmatrix}_{\frac{n}{2}\times n}$$

The prolongation operator and the next matrix of hierarchy, in the corresponding level, is defined in the usual form [12],

$$P_{j-1}^j=P_{\frac{nh}{2}}^{nh}=(R_j^{j-1})^T$$

According to the DWT theory, the measurement of the analyzed matrix must be an integer power of **2** [13]. In the algebraic multigrid (AMG) background this constraint would symbolize a great restriction on the use of the technique. As a purpose of that, it was made the following simplification, the measurement of the restriction operator R_j^{j-1}, for all j, was defined as being $(N/2)\times N$, where

$$N=\begin{cases} n & if \quad n \quad is \quad even, \\ n+1 & if \quad n \quad is \quad odd \end{cases} \tag{2.1}$$

with $n=\dim(A^j)$. If n is odd, coefficient matrix and right hand side vector is used as given is (2.2) and (2.3).

$$\left[A^j\right]_{(n+1)\times(n+1)}=\begin{bmatrix} \left[A^j\right]_{n\times n} & \\ & 1.0 \end{bmatrix} \tag{2.2}$$

$$\left[f^j\right]_{(n+1)\times(n+1)}=\begin{bmatrix} \left[f^j\right]_{n\times n} & \\ & 1.0 \end{bmatrix} \tag{2.3}$$

Therefore, the Algebraic wavelet multigrid (WAMG) setup phase depends only of the choice of the filters coefficients. This approach is very interesting mainly because it avoids the coarsening process and the heuristic parameters present in the standard AMG, simplifying the use of the method as well as its parallel implementation in distributed memory computers [14, 15]. Moreover, the WAMG setup time is powerfully reduced, since it is not needed to calculate the full wavelet transform in each level and the DWT is imperfect to the calculation of a sequence of approximation coefficients. The representation of Haar wavelet via lifting form presented as;

Decomposition:

Consider approximate solution $S = u$ like as signal, and then apply the HWLS decomposition (finer to coarser) procedure as,

$$\begin{aligned} d_1 &= S(2j) - S(2j-1), \\ s_1 &= S(2j-1) + \frac{1}{2} d_1, \\ S_1 &= \sqrt{2}\, s_1 \quad \text{and} \\ D &= \frac{1}{\sqrt{2}} d_1 \end{aligned} \tag{3.3}$$

Reconstruction:

Now apply the HWLS reconstruction (coarser to finer) procedure as,

$$\begin{aligned} d_1 &= \sqrt{2}\, D, \\ s_1 &= \frac{1}{\sqrt{2}} S_1, \\ S(2j-1) &= s_1 - \frac{1}{2} d_1 \quad \text{and} \\ S(2j) &= d_1 + S(2j-1) \end{aligned} \tag{3.4}$$

which is the necessary resolution of the given equation.

4. HAAR WAVELET FILTERS MULTIGRID METHOD FOR THE SOLUTION OF NON-LINEAR BURGURS EQUATION

In this section, we introduce Haar wavelet filters Multigrid method for the solution of Non-linear Burgurs equation. This method uses the values of the function and its derivatives at consecutive points. In order to find numerical solution of the Burgurs equations, it should be discretized in both space and time. The spatial region $[a,b]$ is discretized by N equidistant points with space step $h = x_{i+1} - x_i$, $i = 1, \ldots, N-1$, where $a = x_1 < x_2 < \cdots < x_N = b.$ We consider the following initial boundary value problem,

$$U_t + U U_x = \mu U_{xx}, \quad 0 < x < 1, \quad 0 < t < T \tag{4.1}$$

Coefficient $\mu > 0$ represents the kinematics viscosity of the liquid [3, 16, 17]. With the given conditions:

$$\begin{aligned} &U(x,0) = \varphi(x), \quad 0 < x < 1, \\ &U(0,t) = g_1(t), \quad U(1,t) = g_2(t), \quad t > 0 \end{aligned} \tag{4.2}$$

To set up the HWFMG solution to this problem, let $\mathbf{v}^h$ consist of the grid points $x_j = \dfrac{j}{n}$, for some positive even integer n, and let $u_j = u(x_j)$ and $f_j = f(x_j)$, for j = 0, 1, ….,n. $\Delta t = (0,1]/N$ & $t_s = (s-1)\Delta t, s = 1, 2, \ldots, N.$ Of the many possible ways to discretize this non-linear differential equation, we get the vector equation $A_j^h(u^h) = f^h$ is nonlinear in u^h. To solve this non-linear system of equations, Newton's method can be stated explicitly as follows:

$$u_{n+1} = u_n - J(u)^{-1} F(u), \quad 1 \le n \le N-1 \tag{4.3}$$

To examine the HWFMG correction step, assume that an approximation v^h has been obtained initially on the finer grid, secondly applied the lifting algorithm explained in section 3 for denoising the approximate solution and then residual equation is given by

$$B_j^{nh}(v^{nh} + E^{nh}) - B_j^{nh}(v^{nh}) = r_j^{nh}, \tag{4.4}$$

where, $B_j^{nh} = A^h(u^h) - f^h = 0$ &

Solving the component form of eqn. (4.4), using eqn (4.3) & v^h to obtain **E,** and then substitute this in eqn. (4.4) to obtain r_j^{nh}. Restrict the present approximation and its finer grid residual to the coarse grid successively using

$$r^{\frac{nh}{2}} = R_{nh}^{\frac{nh}{2}} r_j^{nh}, n = 2^j, \; j = J, J-1, J-2\ldots\ldots \tag{4.5}$$

Solve the coarser grid problem $A^{\frac{nh}{2}} e^{\frac{nh}{2}} = r^{\frac{nh}{2}}$ at the desired coarsest level to obtain,

$$e^{\frac{nh}{2}} = \left(A^{\frac{nh}{2}} \right)^{-1} r^{\frac{nh}{2}}$$

Interpolate the error approximation up to the finer grid, successively using $e^{nh} = P_{\frac{nh}{2}}^{nh} e^{\frac{nh}{2}}$,

$$n = 2^j \; j = \frac{J}{2}, \frac{J}{2}+1, \frac{J}{2}+2...... \qquad (4.6)$$

Correct the current finer grid approximation:

$$u^{nh} = v^{nh} + e^{nh}. \qquad (4.7)$$

In the next section, numerical example is studied to demonstrate the accuracy and efficiency of the proposed method.

5. METHOD OF IMPLEMENTATION

In this section, one example is provided to illustrate the validity and effectiveness of the proposed method. The computation associated with the example in this paper is performed using MATLAB. We account norm two of error which is defined by,

$$\|E\|_2 = \left(\sum_{i=1}^{n} (\hat{U}(x_i,t) - U(x_i,t))^2\right)^{\frac{1}{2}}$$

where $\hat{U}(x_i,t)$ is the solution obtained by equation (4.7) solved by HWFMG method and $U(x_i,t)$ is the exact solution.

Consider the Burger's equation [18] with the initial condition,

$$U_t + UU_x = \mu U_{xx}, \quad 0 < x < 1, \; 0 < t < T \qquad (5.1)$$

$$\varphi(x) = U(x,0), \quad 0 \le x \le 1 \qquad (5.2)$$

and non-homogeneous boundary conditions,

$$g_1(1) = U(0,t), \quad g_2(t) = U(1,t), \quad 0 \le t \le T = 1 \qquad (5.3)$$

The exact solution of the Eqn. (5.1) is:

$$U(x,t) = \frac{0.1e^{-A} + 0.5e^{-B} + e^{-C}}{e^{-A} + e^{-B} + e^{-C}}, \; where, \; A = \frac{0.05}{\mu}(x - 0.5 + 4.95t), \qquad (5.4)$$

$$B = \frac{0.25}{\mu}(x - 0.5 + 0.75t) \; and \; C = \frac{0.5}{\mu}(x - 0.5 + 0.375t)$$

Finite difference approximation of equation (5.1) is,

$$A_j^h(u) = \frac{u_j^{nh} - u_j^{(n-1)h}}{k} + u_j^{nh}\left(\frac{u_j^{nh} - u_j^{(n-1)h}}{h}\right) - \mu\left(\frac{u_{j-1}^{nh} - 2u_j^{nh} + u_{j+1}^{nh}}{h^2}\right) = 0, \quad 1 \le j \le N-1 \qquad (5.5)$$

Which is of the form $A^h(u^h) = f^h$ is nonlinear in u^h, solving the nonlinear system of eqn. (5.5), using Newton's Method, $u_{n+1} = u_n - J(u)^{-1}F(u), \quad 1 \le n \le N-1$.

Approximation v^h has been obtained on the finer grid. Residual equation is given by,

$$B_j^{nh}(v^{nh} + E^{nh}) - B_j^{nh}(v^{nh}) = r_j^{nh}, \text{ where } B_j^{nh} = A^h(u^h) - f^h = 0 \qquad (5.6)$$

Appears in component form as,

$$r_j^{nh} = \left(\frac{v_{j+1}^{nh} - v_{j-1}^{nh}}{4k} + \frac{E_{j+1}^{nh} - E_{j-1}^{nh}}{4k}\right) + (v_{j+1}^{nh} + E_{j+1}^{nh})\left(\frac{v_{j+1}^{nh} - v_{j-1}^{nh}}{4h} + \frac{E_{j+1}^{nh} - E_{j-1}^{nh}}{4h}\right)$$
$$- \mu\left(\frac{(v_{j-1}^{nh} + E_{j-1}^{nh}) - 2(v_j^{nh} + E_j^{nh}) + (v_{j+1}^{nh} + E_{j+1}^{nh})}{4h^2}\right)$$

Canceling terms leaves the coarse-grid equation for the unknowns E_j^{nh} : (5.7)

$$\left(\frac{v_{j+1}^{nh}-v_{j-1}^{nh}}{4k}+\frac{E_{j+1}^{nh}-E_{j-1}^{nh}}{4k}\right)+(v_{j+1}^{nh}+E_{j+1}^{nh})\left(\frac{v_{j+1}^{nh}-v_{j-1}^{nh}}{4h}+\frac{E_{j+1}^{nh}-E_{j-1}^{nh}}{4h}\right)-$$
$$\mu\left(\frac{\left(v_{j-1}^{nh}+E_{j-1}^{nh}\right)-2\left(v_{j}^{nh}+E_{j}^{nh}\right)+\left(v_{j+1}^{nh}+E_{j+1}^{nh}\right)}{4h^2}\right)=I_h^{nh}\left(f_j^{nh}-A_j^{nh}(v^{nh})\right)=r_j^{nh}$$

As before, the terms $v_j^{nh}, v_{j+1}^{nh}, v_{j-1}^{nh},$ and r_j^{nh} are obtained by restriction from the finer grid.

Solving Eqn. (5.7) again using Newton's method, we obtain E. Substituting this value of **E** in Eqn. (5.6), we obtain r_{16}^{nh}. Restrict the present approximation and its finer grid residual to the coarse grid successively using,

$$r^{\frac{nh}{2}}=R_{nh}^{\frac{nh}{2}}r_{16}^{nh}, \quad n=2^j,\ j=J,J-1,J-2......$$

Solve the coarse-grid problem $A^{\frac{nh}{2}}e^{\frac{nh}{2}}=r^{\frac{nh}{2}}$ at the desired coarsest level to obtain,

$$e^{\frac{nh}{2}}=\left(A^{\frac{nh}{2}}\right)^{-1}r^{\frac{nh}{2}}.$$

Interpolate the error approximation up to the finer grid, successively using, $e^{nh}=P_{\frac{nh}{2}}^{nh}e^{\frac{nh}{2}}$,

$$j=\frac{J}{2},\frac{J}{2}+1,\frac{J}{2}+2......$$

These corrections are interpolated up to the finer grid and used to update the finer grid approximation v^h.

Correct HWFMG solution of the problem 5.1 with 5.2 is $u^h=v^h+e^h$. Computer simulation was carried out in the case N= 16, the computed HWFMG results are compared with the AMG and exact solution are presented in the Table 5.1 & 5.2. More accurate results can be obtained by using a larger N. In Table 5.3, we show norm two of error for various values of μ, t and N with Δt = 0.0001.

Table 5.1. AMG, HWFMG and Exact Solutions of Problem 5.1 for N=16, t=0.0001.

x	AMG	HWFMG	Exact
0.05882	0.59712	0.59700	0.56336
0.11765	0.59384	0.59383	0.55935
0.17647	0.58962	0.58961	0.55534
0.23529	0.58537	0.58536	0.55133
0.29412	0.58112	0.58111	0.54733
0.35294	0.57687	0.57686	0.54332
0.41176	0.57263	0.57262	0.53933
0.47059	0.56839	0.56838	0.53533
0.52941	0.56416	0.56415	0.53135
0.58824	0.55994	0.55993	0.52737
0.64706	0.55572	0.55572	0.52339
0.70588	0.55151	0.55150	0.51943
0.76471	0.54731	0.54731	0.51547
0.82353	0.54312	0.54311	0.51152
0.88235	0.53891	0.53891	0.50758
0.94118	0.53571	0.53570	0.50365

Table 5.2. AMG, HWFMG and Exact Solutions of Problem 5.1 for N=16, t=0.000001.

x	AMG	HWFMG	Exact
0.05882	0.56368	0.56368	0.56335
0.11765	0.55966	0.55966	0.55934
0.17647	0.55565	0.55565	0.55533
0.23529	0.55164	0.55164	0.55132
0.29412	0.54764	0.54764	0.54732

0.35294	0.54363	0.54363	0.54332
0.41176	0.53963	0.53963	0.53932
0.47059	0.53564	0.53564	0.53533
0.52941	0.53165	0.53165	0.53134
0.58824	0.52767	0.52767	0.52736
0.64706	0.52369	0.52369	0.52339
0.70588	0.51972	0.51972	0.51942
0.76471	0.51576	0.51576	0.51546
0.82353	0.51181	0.51181	0.51151
0.88235	0.50787	0.50787	0.50757
0.94118	0.50394	0.50394	0.50364

Table 5.3. Norm two of error for various values of μ, t and n of the problem 5.1.

t / μ, n	μ = 1, N = 256	μ = 1, N = 512	μ = 0.1, N = 256	μ =0.1, N = 512
0.2	2.4244E− 11	1.7892E − 10	2.9949E −5	2.7542E − 6
0.4	6.4101E − 11	1.8886E − 10	6.7446E −5	5.1001E − 6
0.6	3.7221E − 10	5.0883E − 10	8.9219E −5	6.9656E − 6
0.8	6.0105E − 10	6.8384E − 10	8.2415E −5	7.3609E − 6
1	6.3600E − 10	7.2012E − 10	5.5745E −5	5.7378E − 6

6. CONCLUSIONS

An efficient haar wavelet filters multigrid method for the numerical solution of non-linear partial differential equation is proposed. From the figures and tables, the proposed scheme is very convenient and effective. However the CPU time of HWFMG method is lower than others. Hence the scheme has expansive variety of applications in science and engineering field.

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