The Simplest Realistic Model of Interaction Potentials in Study of Spiked Harmonic Oscillators

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Abstract

A suitable model of interaction potentials can be used in detailed study of spiked oscillators with complex shift of co-ordinates and non Hermitian super symmetry in a quantum mechanics. The study of spiked harmonic oscillators are used in atomic physics, molecular physics, nuclear physics and particle physics. The formalism and the techniques of SUSY quantum mechanics is generalized to the cases where the super potential is generated/defined by higher excited eigen states. The SUSY formalism applies everywhere between the singularities. A systematic application of the formalism to the other potentials with known spectra would yield an infinitely rich class of "solvable" potentials in terms of their partner potentials.

Key words: Spiked harmonic oscillator, complex shift of co-ordinates and non hermitian supersymmetry.

1. Introduction:

Now a days, study of interesting feature of quantum field theory using reduction of space time to a point takes place [4]. In its turn, the quantization recipe becomes non unique. Such a type of ambiguity was unexpected. A certain complexified version of quantum mechanics [7] obtained using one dimensional space time continuum. In such a slightly more realistic setting the anbiguity results from the indeterminate complex asymptotic boundary conditions [3, 12].

Within this fresh methodical framework, there is detailed study of traditional concept of quantum mechanics [1,7]. In an attempt to apply the PT symmetric formalism of the nonhermitian quantum mechanics of more particles [3,8,9], we were forced to return to the general spiked harmonic oscillator. This provided a key motivation for our forthcoming discussion. The PT symmetric approach of the field theory opens new unresolved questions. The Witten's super symmetry in quantum mechanics [1, 5, 6] did also find its natural PT symmetrized new versions [12]. An explicit demonstration of their applicability to the exactly solvable spiked harmonic oscillators is still missing and will in fast be a core of our present work.

2. SPIKED OSCILLATORS:

Consider the Hamiltonian of the system with a potential containing parabolic term x^2, describing ordinary quantum oscillator and a singular term $\frac{1}{x^{\mu}}$ have a sharp spike at $x = 0$.

A typical Hamiltonian of spike harmonic oscillators is given by

$$H = \frac{p^2}{2m} + V \qquad \dots\dots..(1)$$

Since $\vec{p} \to -i\hbar\vec{\nabla}$ *and* $V = Potential\ energy$

For one dimensional motion, $p \to -i\hbar\frac{\partial}{\partial x}$

$$\Rightarrow H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad \dots\dots\dots\dots(2)$$

For convenience, take $\hbar = 1 = 2m$

Then we have $H = -\frac{d^2}{dx^2} + V \qquad \dots\dots\dots(3)$

For an ordinary oscillators, $V = x^2 \qquad \dots\dots\dots(4)$

and for spiked harmonic oscillators,

$$V = x^2 + \frac{\lambda}{x^\mu} \qquad \dots\dots\dots(5)$$

Here $\lambda = constant\ and\ 0 \le x \le \infty$

The study of spiked harmonic oscillators consists two parts:

[A] COMPLEX SHIFT OF COORDINATES: A specific continuation of the wave function with a definite parity [13] being used and for an operator P, we can write

$$P\Psi_\pm(r) = \Psi_\pm(-r) = \pm\Psi_\pm(r) \qquad \dots\dots(6)$$

Here $\Psi_\pm$ = Wave function

In equation (6), such a wave function $\Psi_\pm$ is non analytic. This type of regularization was proposed in a slightly different context by Buslaev and Grecchi [14]. Its present implementation has been thoroughly discussed in [3, 10]. It is equivalent to parallel imaginary shift of the real line of co-ordinates and is given by –

$$r = r(x) = (x - i\varepsilon) \qquad \dots\dots(7)$$

Where $x\epsilon\ (-\infty, \infty)$

It form preserves the so called P T symmetry such that

$$PT\ r(x) = r(-x)$$

with parity P and T. $i = -i$ with complex conjugation T.

In the broader anharmonic oscillator context the appeal of PT symmetry has been interpreted by Bender and Boettcher [7] as a substitute for the lost Hermiticity after complexification.

Within the PT symmetric version of quantum mechanics [15], the study of harmonic oscillator equation

$$\frac{-d^2}{dr^2}L(r) + \frac{\left(\mu^2 - \frac{1}{4}\right)}{r^2}L(r) + r^2 L(r) = E\ L(r) \qquad \dots\dots\dots(8)$$

Here μ = Positive real number

& L(r) = Laguerre's polynomial

that all the PT symmetric eigenstates of H^μ are explicitly defined by the equation i.e energy and is given by

$$E = E_n^{\beta} = (4n + 2\beta + 2) \qquad \text{.......(9)}$$

Where $n =$ degree of Laguerre's polynomial

and $\beta =$ order of Laguerre's polynomial

provided that $\beta = \pm\mu$ ....(10)

Such supports the regularization given by equation (7) as preserving a maximal similarity to the one dimensional regular oscillator. The selection rules associated with equation (7) lies in the significant facilitation of super symmetrization.

(B) Non Hermitian Super Symmetry:

Consider the regularized harmonic oscillator wave functions with zero subscript which give the two alternative auxiliary Witten's superpotentials [11] and is given by

$$W^{\delta}(r) = -\frac{\partial_r L_0^{\delta}(r)}{L_0^{\delta}(r)} = r - \frac{\left(\delta + \frac{1}{2}\right)}{r} \qquad \text{......(11)}$$

Where $\delta = \pm\mu$

There use leads to $\delta -$ numbered set of the partner harmonic oscillator. The Hamiltonian pairs are denoted by H_L^{δ} *and* H_R^{δ} and are given by

$$H_L^{\delta} = \hat{p}^2 + \left[W^{\delta}\right]^2 - \left[W^{\delta}\right]^1 \qquad \text{.......(12)}$$

and

$$H_R^{\delta} = \hat{p}^2 + \left[W^{\delta}\right]^2 + \left[W^{\delta}\right]^1 \qquad \text{........(13)}$$

After appropriate insertions, we have common compact form as

$$H_L^{\delta} = H^{\propto} - 2\delta - 2 \text{ and } H_R^{\delta} = H^{\overline{\propto}} - 2\delta \qquad \text{.........(14)}$$

Where $\overline{\propto} = |(\delta + 1)|$

This implies the existence of the four series of energies, given by

$$E_L^{+} = 4n;\ E_L^{-} = 4n - 4\delta\ ;\ E_R^{+} = 4n + 4 \text{ and } E_R^{-} = 4n - 4\delta \qquad \text{.........(15)}$$

associated with the respective wave functions,

$$L_n^{\delta},\ L_n^{-\delta},\ L_n^{\delta+1} \text{ and } L_n^{-(\delta+1)} \text{ where } n = 0, 1, 2, 3, \ldots\ldots$$

As expected, the zero energy ground state (with $E_L^{+} = 0$) is exceptional as unmatched by any $R -$ subscripted partner.

The first non trivial example of the generalized super symmetric partnership can be used on the choice of the positive $\delta = \frac{1}{2}$. As an illustration of the breakdown of the super symmetric isospectrality for singular superpotentials in Hemitian setting. The PT symmetry enables us to reintroduce the missing levels in a way displayed in Table 1.

TABLE 1: PT SUSY Between H_L and H_R

$E_{L/R}$	$A^{\frac{1}{2}}$ $\|n_L\rangle$ ------------------------→ $\|n_R\rangle$
…….	………. ……….. …..
…….	………. ……….. …..
10	$L_3^{-\frac{1}{2}} -------------\rightarrow L_3^{-\frac{3}{2}}$
8	$L_2^{\frac{1}{2}} -------------\rightarrow L_1^{\frac{3}{2}}$
6	$L_2^{-\frac{1}{2}} -------------\rightarrow L_3^{-\frac{3}{2}}$
4	$L_1^{\frac{1}{2}} -------------\rightarrow L_0^{\frac{3}{2}}$
2	$L_1^{-\frac{1}{2}} -------------\rightarrow L_1^{-\frac{3}{2}}$
0	$L_0^{\frac{1}{2}} -------------\rightarrow 0$
-2	$L_0^{-\frac{1}{2}} -------------\rightarrow L_0^{-\frac{3}{2}}$

Our present type of regularization looks very natural in this comparison. We may notice the following few unusual features of our nonstandard and innovative implementation of the PT super symmetric pattern.

It can only observe that the "unmatching" involves the ground state again. It is separated only the double distance and possesses a negative superscript as shown in Table 2.

TABLE 2: Reversal of Table 1

$E_{L/R}$	$A^{-\frac{3}{2}}$ $\|n_L\rangle$ ------------------------→ $\|n_R\rangle$
…….	………. ……….. …..
…….	………. ……….. …..
14	$L_2^{\frac{3}{2}} -------------\rightarrow L_2^{\frac{1}{2}}$
12	$L_3^{-\frac{3}{2}} -------------\rightarrow L_2^{-\frac{1}{2}}$
10	$L_1^{\frac{3}{2}} -------------\rightarrow L_1^{\frac{1}{2}}$
8	$L_2^{-\frac{3}{2}} -------------\rightarrow L_1^{\frac{1}{-2}}$
6	$L_0^{\frac{3}{2}} -------------\rightarrow L_0^{\frac{1}{2}}$
4	$L_1^{-\frac{3}{2}} -------------\rightarrow L_0^{-\frac{1}{2}}$
2	…… ……… ………….
0	$L_0^{-\frac{3}{2}} -------------\rightarrow 0$

3. Results and Discussion:

In this research paper, I shall try to implant a certain more satisfactory mathematical symmetry into the set of harmonic oscillator wave functions. What can we expect from moving to larger semintegers, $\propto = |\delta|$? Just a strengthening of the tendencies which were revealed in Table 1 and 2. There features can be simply extrapolated. Thus, a $\delta = \frac{3}{2}$ modification of table 1 will contain two more lines at its bottom. At the next positive, $\delta = \frac{5}{2}$ supersymmetry between $H_L = H^{5/2} - 7$ and $H_R = H^{\frac{7}{2}} - 5$ will introduce a ground state mapping i.e. $L_0^{-\frac{5}{2}} \xrightarrow{A^{\frac{5}{2}}} L_0^{-\frac{-7}{2}}$. It appears at $E_{\frac{L}{R}} = -10$, witnessing just a continuing downward shift of the levels with the negative and decreasing superscripts.

Upto a constant shift, the PT supersymmetric partners coincide is given by the solution of the algebraic equation $|\delta| = |\delta + 1|$. This solution is unique $\left(for\ \delta = -\frac{1}{2}\right)$ and corresponds to the case where the poles in $A^{\delta}\ and\ B^{\delta}$ vanish. This is the only case tractable also without the use of the PT regularization.

A very exceptional role is played by the integers limits of $\propto$. In contrast, no special attention must be paid to the limit of vanishing spike $\left(\propto \rightarrow \frac{1}{2}\right)$. In general, with $\propto \neq \frac{1}{2}$, our PT regularization $(\in \neq 0)$ can also be removed, if needed, via the limiting transition $(\in \rightarrow 0)$ accompanied by the necessary halving of the axis of co-ordinates. This means that we have to replace $r(x) = (x - i \in)$ by the radial and real $r \in (0, \infty)$ and cross out all the states with $\beta < 0$. They are simply proclaimed "in acceptable" in the light their conventional interpretation.

4. Conclusion:

Harmonic oscillators with a centrifugal spike are analyzed via a non Hermitian regularization, within a compflexified SUSY quantum mechanics. In this research paper we conclude that the normalizable ground state exists for both $H_L\ and\ H_R$ such that

$$E_{L0}^{-} = E_{R0}^{-} = -2$$

and the first excited state remains unmatched by any R-subscripted partner at

$E_L^{+} = 0\ as\ excepted.$

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