Study of Dissipativity Criteria for Digital Filters with Saturation Nonlinearity

Rajeev Kumar[1], Siva Kumar Tadepalli[2]
[1,2]*Dept. of Electronics and Telecommunication, BIT DURG, CG*
[1]rajeevicesrm@gmail.com

Abstract

This paper describes the concepts of dissipative dynamical systems and illustrates the dissipation inequality with storage functions and supply rate functions. This paper describes how dissipativity can be used to ensure the stability of dynamical systems. This paper also reveals the problems associated with a proper selection of storage function since it is a range of possible storage. The concept has been further applied to ensure the dissipativity criteria of digital filters of the direct form with saturation non-linearity. This paper further investigates whether the interconnected direct form of digital filters is also dissipative and asymptotically stable.

Keywords: *dissipativity, storage functions, digital filters, direct form, interconnection, stability.*

1. Introduction:

In control systems we deal not only with stationary systems but dynamical systems as well. The interpretation of dynamical systems is not so easy to deal with. At any given time a dynamical system has a state which keeps changing as the input is supplied or with given external disturbances. Analysis and synthesis are the two main aspects of control systems. In analysis we analyze the performance of the system with given input and disturbances. In synthesis we design a controller to meet the required performance. There are many such tools from classical control to modern control theory to meet these two aspects.

However the concept of dissipativity which was introduced by J.C. Willlems (1972) [1] is a unified framework for the analysis and synthesis of complex dynamical systems. The concept of dissipativity is commonly applied among physics, systems theory and control engineering. The purpose paper is to review the basic concepts of dissipativity and its applications in control and filtering.

The basic concepts of dissipativity were further extended to linear systems with quadratic supply rates [2]. This concept has been generalized by Hill and Moylan (1980) [3] [4].Dissipative controllers were also studied for continuous [5] and discrete time systems [6]. Further the concepts of dissipativity have been extended for stochastic systems with time delay [7]. In recent years dissipativity and its applications have been studied for static neural networks with time delay [8] and fuzzy time -delayed systems [9].In this review paper we focus on the basic concepts of dissipative systems and also how the dissipativity can be used.

While designing digital filters based on digital hardware or software model one has to divide it into small digital filters before hardware or software implementation. Due to this reason there is interference or disturbances between the small divided filters which leads to unstable filters and the performance also gets affected [10] [11]. C.K. Ahn handled this problem of instability, interferences and disturbances and proposed new stability results for interfered one dimensional (1-D)systems [12], [13] and two dimensional (2-D) systems [14]- [19].

However work by C.K. Ahn only analyzed single fixed point digital filters and it did not analyzed digital filters of direct form and it also did not consider the interconnected filters. So this paper [20] reviews on dissipativity criteria of digital filters of the direct form with saturation non- linearity and also investigates whether the dissipativity of the interconnected digital filters can be ensured. Once the dissipativity criteria for the direct form of digital filters is obtained it can be generalized and extensively used for the stability analysis and synthesis of controllers with exogenous inputs and disturbances [20]

This review paper is organized as follows:

(i)Section 2 reviews on the basic concepts and mathematical model associated with dissipativity and stability of dissipative systems along with some definitions.

(ii) Section 3 reviews on the dissipativity criteria for digital filters of the direct form with saturation non-linearity and also investigates the dissipativity of interconnected digital filters.

(iii) In section 4 some comparisons and analysis regarding various aspects of dissipativity have been done.

Finally some conclusions are drawn regarding the overall review.

2. Basic concepts of Dissipativity: Dissipativity is a unified framework for the analysis and synthesis of control and filtering problems. This section describes the basic concepts of dissipativity.

2.1 Dissipative Systems:

Dissipative systems can be defined as thermodynamically open systems which can interchange matter and energy with the surroundings. The term dissipation means loss of energy in abstract manner. Examples include electrical circuits in which electrical energy is dissipated in the form of heat energy to the resistors. Similarly when a block is moved on the rough surface then mechanical energy is lost in the form of frictional heat. In viscoelastic systems, viscous force is responsible for dissipation or loss of energy. For example honey, when it flows, it resists shear flow and strain when stress is applied.

Thermodynamic systems for which second law describes a form of dissipation leading to increase in entropy [1] are also examples of dissipative systems. Entropy means a measure of disorder in the system. Entropy $\boldsymbol{S}$ may be described in terms of macroscopic configuration $\boldsymbol{\Omega}$ that a thermodynamic system can have with a given state.

$$\boldsymbol{S} = k_B\ \boldsymbol{\Omega}$$

Where k_B: Boltzmann constant

If we put gas in a container with known volume, pressure and energy then it will have a very large number of possible molecular configurations. At equilibrium this configuration is not unique. Entropy may be understood as the measure of disorder within a macroscopic system. Second law of thermodynamics states that isolated system entropy always increases as time passes and it finally moves towards thermodynamic equilibrium, the state with a maximum entropy or disorder. Non –isolated or open systems may lose entropy provided the environment entropy increases by at least that amount.

It should be understood that Entropy is a function of state of the system and a change in entropy of the system is determined by its initial and final states. Entropy refers to the stored energy of systems.

2.2: Storage functions: At any given time or at any state a system have some energy. If some external input energy is supplied then the system can be shifted to a more energetic state.

Storage functions are the basic internal properties of the systems that give the knowledge of behaviour of systems. Storage functions are easy to describe but not so easy to determine. They are not just uniquely determined by the input/output behaviour of the systems [1]. Storage functions associated with dynamical systems satisfy an inequality: it is bounded from below by the available storage and bounded from the above by required supply. So there is a continuous range of possible storage functions from lower to upper bounds [1].

$$\boldsymbol{s_a(x) \leq s(x) \leq s_r(x)}$$

$s_a(x)$: available energy

$s_r(x)$: required supply

Available storage of a system is the amount of internal storage which may be extracted from the system and the required supply is the amount of supply which has to be given to the system in order to shift it from minimum storage to a given state.

In the case of viscoelasticity input/output behavior can be understood in terms of relaxation functions [1]. However the knowledge of internal storage functions cannot be just obtained by the relaxation functions alone but require additional information about complex physical process associated with the system. Such an input/output description is starting point of the systems with memory. The state space analysis has become an important tool to overcome this difficulty [1], [2].

2.3 Dissipative Dynamical Systems:

A Dynamical system $\sum$ is said to be dissipative with respect to supply rate **w** if there exist a non-zero storage function **S** called storage function such that for all $(\boldsymbol{t_1}, \boldsymbol{t_0}) \in \boldsymbol{R_2}^+$,$x_0 \in \boldsymbol{X}$and $u \in U$,we have:

$$S(x_0) + \int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}} w(t)dt \geq S(x_1)$$

Where$x_1 = \phi(t_1, t_0, x_0, u)$:state-transition function ;$w(t) = w\big(u(t), y(t)\big)$: Input or supply rate function ; $y = y(t_0, x_0, u)$: output of dynamical system [1].

Inequality described above is called dissipation inequality.

We note that:

$$\oint w(t)dt \geq 0$$

Where the circular integration indicates that the dynamical system is shifted from initial state to a terminal state along some path in state space.

It is also clear from the dissipation inequality that storage at a given time cannot exceed the storage at time zero added with the supply given for that interval. Hence the dissipative systems either dissipate energy or store energy. They do not have their own energy.

The method taken here proceeds from the knowledge from physical considerations that the dynamical system is dissipative and the storage function exists. The fact that the storage function is defined by just the dissipation inequality requires further analysis.

2.4 Stability of Dissipative systems:

Dissipativity is a generalization of the concept of passivity. A feedback system which contains passive dynamical system in both forward and feedback loop is itself passive and thus stable. Also it can be said that, moreover the summation of stored energy in both the forward loop and the feedback loop is a Lyapunov function for the closed loop system [1].

As the isolated systems are far away from environment so clearly there is no source of input energy to the system so the dissipation inequality reduces to the fact that the storage function is a Lyapunov function. However for the open systems which interact to the environment there may be some exogenous input to the system so in this case dissipation inequality holds and we analyze the stability based on dissipativity of the system.

Some technical conditions should meet in order for dissipativeness to imply stability of equilibrium at a local minimum of storage function [1].These are:

(i) The system is isolated i.e. input state should contain just one element and should be constant $u(t) = u^*$

(ii) x^* is an equilibrium point. It is the only equilibrium point as time passes.

(iii) X is a subset of normed space.

(iv) $x_1 = \phi(t_1, t_0, x_0, u^*)$,the state transition function is continuous in t as time passes.

(v) $w\big(u^*, r(x, u^*)\big) \leq 0$ forall x in the neighbourhoodof x^*i.e the supply rate to the system should not exceed zero.

The following definition of stability is a standard one in the context of Lyapunov|:

Definition [1]: *The equilibrium point* x^* *of dynamical system is said to be stable if for given ε>0 there exist δ (ε)>0 [1].*

Such that$\|x_0 - x^*\| \leq \delta$ *implies*

$$\|\phi(t_1, t_0, x_0, u^*) - x^*\| \leq \varepsilon \quad for\ all\ t \geq t_0$$

This definition describes that if a state is near equilibrium initially and the state transition function continues to remain close to the equilibrium as time passes then equilibrium point is said to be stable.

A very useful method for analyzing stability is by means of Lyapunov functions [1]. The description of Lyapunov function is introduced in the following definition.

Definition *[1]*: *A real valued function V defined on the state space X of dynamical system is said to be a Lyapunov function in the neighborhood of the equilibrium point x^* if:*

(i) V is continuous atx^*.

(ii) V attains a strong local minimum at x^*.

(iii) V is monotonic non-increasing along solutions in the neighborhood ofx^*.

An equilibrium point x^* is stable if there exist a Lyapunov function in the neighborhood of x^*. If the storage function S has all the properties of Lyapunov function V described earlier then it may lead to the following conclusion:

An equilibrium point x^* of a dissipative dynamical system is stable if the storage function S is continuous and attains a strong local minimum at point x^*. Also S is monotonic non increasing along the solutions in the neighborhood of x^* .Moreover S is a Lyapunov function in the neighborhood of x^*

2.5 Interconnected systems:

The analysis of stability of dynamical systems can be done effectively with the help of the concept of dissipativity, however there may be some problems regarding this.

One has to choose a proper supply rate and also the selection of stored energy function is also crucial since there is a range of possible storage function from lower bound to upper bound i.e. from available storage to required supply. So if we choose anyone as the storage function then it may lead to the variational problems. Especially in the case of non-linear systems this type of problems are very difficult to overcome.

The concept of interconnected systems is a very effective tool to overcome this problem. It allows one to construct a storage function that is neither available storage nor required supply and only can be constructed by solving variational problems. It can be shown that if a dissipative system is divided into many subsystems which are interconnected to each other then the number of possible storage functions is greatly reduced.

3. Direct form of digital filters with saturation non-linearity:

Based on the concept of abstract energy exchange between systems and surroundings the theory of dissipativity can also be applied for stability analysis of digital filters [20].

Consider transfer function of the digital filter:

$$G(z) = h_0 z^{-n} + h_1 z^{-(n-1)} + h_2 z^{-(n-2)} + \cdots + h_n \qquad (1)$$

With saturation non linearity given as:

$$f(y(r)) = \begin{cases} 1 \ , if \ y(r) > 1 \\ y(r) \ , if \ -1 \leq y(r) \leq 1 \\ -1 \ , if \ y(r) < -1 \end{cases}$$

This satisfies the following properties:

$$f(0) = 0, \ and \ 0 \leq \frac{f(y(r))}{y(r)} \leq 1 \quad (2)$$

The digital filter (1) can also be expressed in simplified form as:

$$x(r+1) = Ax(r) + Bfy(r) + Bu(r) \quad (3)$$

$$y(r) = H^T x(r) + h_n u(r) \quad (4)$$

Where A, B, H and u(r) are matrices of appropriate dimensions

$y(r)$*: Output signal of* $G(z)$

$f(y(r))$+u(r)*: input signal of* $G(z)$

$u(r)$*: External input*

3.1Dissipativity criteria of direct form of digital filters with saturation non-linearity:

For $\alpha \geq 0$ and scalars Q, S and R digital filter (3)-(4) is (Q, S, R) –α dissipative if:

$$\sum_{r=0}^{T} Q\, y^2(r) + 2\sum_{r=0}^{T} Sy(r)u(r) + \sum_{r=0}^{T} Ru^2\,(r) \geq \alpha \sum_{r=0}^{T} u^2\,(r) \quad (5)$$

under the zero initial condition where *T>0*

Where α: dissipativityPerformance bound

A novel (Q, S, R) -α dissipativity criteria for the direct digital filters (3)-(4) is given as following [20]

Theorem 1[20]*: Digital filter (1) is (Q,S,R)-*α *dissipative if for given* $\alpha \geq 0$ *there exist a matrix variable* $P = P^T > 0$ *and scalar variables* $\delta > 0$ *and* $m > 0$

Such that theLMI,

$$\Gamma = \begin{bmatrix} \Gamma_{1,1} & * & * \\ \Gamma_{2,1} & \Gamma_{2,2} & * \\ \Gamma_{3,1} & \Gamma_{3,2} & \Gamma_{3,3} \end{bmatrix} < 0 \quad (6)$$

Where the matrix terms

$\Gamma_{1,1} = A^T PA - P - QHH^T + \delta HH^T$

$\Gamma_{2,1} = B^T PA + mH^T$

$\Gamma_{2,2} = B^T PB - 2m - \delta$

$\Gamma_{3,1} = B^T PA - Qh_n H^T - SH^T + \delta h_n H^T$

$\Gamma_{3,2} = B^T PB + mh_n$

$*$: *Symmetrical terms*

The proof theorem 1of how this LMI ensures dissipativity is analyzed in [20].

3.2 Precise form of Dissipativity:

The free weighing matrix approach presented in [13] can be used to obtain dissipativity with a better performance bound. It improves potential conservatism of LMI (6).

If use the term obtained by (3) and multiply a new term including matricesN_1, $N_2$and N_3[13]

We have:

$$2[x^T(r)N_1 + x^T(r+1)N_2 + u^T(r)N_3]$$

$$[Ax(r) + Bf\big(y(r)\big) + Bu(r) - x(r+1)] = 0 \quad (7)$$

Using this term in previous theorem the LMI expands to: $\breve{\Gamma}$ a 4x4 matrix [20].

Similar to the theorem 1

$\breve{\Gamma}$<0 ensures improved crtiteria (Q,S,R)- α dissipativity of digital filters (3) and (4)with better performance.

3.3Dissipativity of direct form of interconnected filters:

The (*Q*,*S*,*R*)- α dissipativity is a useful tool for analyzing dissipativity criteria of interconnected digital filters[20]

Consider a digital filter represented by an interconnection of (*(Q*,*S*,*R*)-*α* dissipative digital sub - filtersDD_k in the direct form where

DD_k:

$$\begin{cases} x_k(r+1) = Ax_k(r) + Bf\big(y_k(r)\big) + Bu_k(r) \\ y_k(r) = {H_k}^T x_k(r) + h_{k,n} u_k(r), 1 \le k \le m \end{cases}$$

Where

$x_k(r)$: State vector signal of DD_k

$u_k(r)$: External input signal of DD_k

$y_k(r)$: Output signal ofDD_k

Theorem 2[20]: *Given the scalars* $\alpha \geq 0$, Q_k , S_k , R_k *assume that there exists a matrix variable* $P_k = P_k^T > 0$*and scalar variables* $\delta_k > 0$ *and* $m_k > 0$ *satisfying the LMI condition for the* DD_k *Where*$1 \leq k \leq m$. *The interconnected digital filter in the direct form is then* $(Q_{IC}, S_{IC}, R_{IC}) - \alpha$ *dissipative where:*

$$Q_{IC} = \overline{Q} - 2\overline{S}\,\Omega + \Omega\,\overline{R}\Omega - \alpha\Omega^2$$

$$S_{IC} = \overline{S} - \Omega\,\overline{R} - \alpha\Omega \quad and \quad R_{IC} = \overline{R} - \alpha I$$

$$\Omega = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

$$\overline{Q} = diag\{Q_1, Q_2, \ldots\} \quad \overline{S} = diag\{S_1, S_2 \ldots.\} \overline{R} = diag\{R_1, R_2 \ldots.\}$$

The disspativity criteria for interconnected systems are similar to (5). Further for the asymptotic stability of interconnected direct form of digital filters we have to proceed with unforced systems i.e. systems with zero input.

4. Analysis and comparisons:

Here are some of the comparisons which are inferred from this review:

(i) Before hardware or software implementation the digital filters must be divided into small ones which lead to interferences. This problem was eliminated by C.K Ahn et. al. [13] who proposed the strict dissipativity of fixed point digital filters in the state space form.

However this work considered only single fixed point digital filters and did not consider the direct form of digital filters and the interconnected digital filters. So the present paper [20] investigates the dissipativity criteria for digital filters of the direct form and also finds whether the interconnected direct for of digital filters are dissipative and asymptotically stable?

(ii) The free weighing matrix approach presented in [13] can be applied to obtain a precise form of dissipativity criteria. It improves the potential conservatism of LMI. Also it gives the several times improved optimal dissipative performance bound.

(iii) In part-I of dissiapative dynamical systems [1] the mathematical model employed was state space model. Storage functions, dissipative inequality, interconnected systems, stability were some of the important concepts discussed.

Part-II [2] presents the theory of dissipative systems in the context of FDLS (finite dimensional linear systems) with quadratic supply rates. Quadratic storage functions have been analyzed in this paper and it has been characterized in terms of certain matrix inequalities.

Further the concepts of dissipativity were generalized to nonlinear systems. Paper [3] describes the conditions for which whether linear systems with certain non-linear feedbacks were stable. Also paper [4] reveals connection between finite gain and asymptotic stability.

(iv) The analysis of dissipativity in this review is of one-dimension. However it can also be generalized for two-dimensional systems [27].Clearly in the practical world we frequently encounter the two-dimensional systems [21]-[26] in which one dependent variable depends on two independent variables. For multidimensional systems we require multidimensional controllers which can handle more than one parameter simultaneously. It is extremely useful to obtain dissipativity criteria for multidimensional systems.

5. Conclusion:

This paper describes the general theory of dissipative dynamical systems. Dissipative systems have been described mathematically in terms of dissipation inequality. This paper reveals the problems associated with the selection of proper storage functions as it is a possible range of storage. This problem can be reduced when we consider interconnected systems. Stability of the dynamical systems which is of major concern has been analyzed with the concept of dissipativity. Finally the results been applied for the dissipativity of direct form of digital filters and it has been found that direct form of digital filters with saturation non-linearity are also dissipative. The interconnected direct form of digital filters is also dissipative and asymptotically stable.

References

[1].J.C. Willems, “Dissipative dynamical systems: I. general theory,” *Archive Rational Mech. Anal*., vol. 45, **(1972)**, pp. 321–351.

[2]. J.C. Willems, “Dissipative dynamical systems –part: 2-Linear systems with quadratic supply rates”, Archive Rational Mech. Anal. vol. 45,(1972), pp. 321–351.

[3].D.J. Hill and P. J. Moylan, “Stability results for non linear feedback systems”, Automatica, vol. 13, (1977), pp. 377-382.

[4].Hill, D. J., & Moylan, P. J, “Connections between finite gain and asymptotic stability” ,*IEEE Transactions on Automatic Control*, 25, (**1980),**pp.931-936

[5]. S. Xie, L. Xie, and C. Souza, “Robust dissipative control for linear systems with dissipative uncertainty,” *Int. J. Control*, vol. 70, no. 2, (**1998**), pp. 169–191.

[6]. Z. Tan, Y. Soh, and L.Xie, “Dissipative control for linear discrete-time systems,” *Automatica*, vol. 35, no. 9, (**1999**), pp. 1557–1564.

[7].H. Zhang, H. Yan, and Q. Chen, “Stability and dissipative analysis for a class of stochastic system with time-delay,” *J. Franklin Inst.*, vol. 347, no. 5, (**2010**),pp. 882–893.

[8].Z. Wu, J. Lam, H. Su, and J. Chu, “Stability and dissipativity analysis of static neural networks with time delay,” *IEEE Trans. Neural Netw. Learning Syst*., vol. 23, no. 2,(**2012**), pp. 199–210.

[9].X. Su, P. Shi, L. Wu, and M. Basin, “Reliable filtering with strict dissipativity for fuzzy time-delay systems,” *IEEE Trans. Cybernetics*, vol. 44, no. 12, (**2014**), pp. 2470–2483.

[10]. Y. Tsividis.,“ Mixed analog-digital VLSI devices and technology”, *world scientific publishing*, (**2002)**.

[11].J.Monteiro and R.V. Leuken, editors, “Integrated Circuits and System Design: Power and Timing Modeling, Optimization and Simulation, *Springer* (**2010)**

[12].C.K.Ahn, “Some new results on stability of direct-form digital filters with finite wordlength non-linearity”, *Signal Processing*, 108 :(**2015**), pp.549-557.

[13]. C.KAhn and P.Shi, “Dissipativity analysis of fixed-point interfered digital filters”, *Signal Processing*, 109: (**2015**), pp.148-153.

[14].C. K. Ahn, “$l2$–$l\infty$ elimination of overflow oscillations in 2-D digital filters described by Roesser model with external interference,” *IEEE Transactions on Circuits Syst. II*, vol. 60, no. 6, (**2013**), pp. 361–365.

[15]. C. K. Ahn, “$l2$–$l\infty$ suppression of limit cycles in interfered two dimensional digital filters: A Fornasini-Marchesini model case,” *IEEE Transactions on Circuits Syst. II*, vol. 61, no. 8,(**2014**) pp. 614–618.

[16]. C. K. Ahn, “Overflow oscillation elimination of 2-D digital filters in the Roesser model with Wiener process noise,” *IEEE Signal Processing Letters*, vol. 21, no. 10, (**2014**), pp. 1302–1305.

[17]. C. K. Ahn, “New passivity criterion for limit cycle oscillation removal of interfered 2-D digital filters in the Roesser form with saturaton non-linearity”, *Nonlinear Dynamics*, 78(1): (**2014**), pp.409-420

[18]. C. K. Ahn and H. Kar, “Expected power bound for two-dimensional digital filters in the Fornasini-Marchesini local state-space model,” *IEEE Signal Processing Letters*, vol. 22, no. 8, (**2015**), pp. 1065–1069.

[19]. C.K Ahn and H.Kar.“Passivity and finite-gain performance for two-dimensional digital filters: The FM LSS model case”, *IEEE Transactions on Circuits systems.-II*, 62(9),(**2015**), pp.1549-7747.

References contd.

[20].C.K Ahn, P.Shi , " Strict dissipativity and asymptotic stability of digital filters in direct form with saturation non-linearity", *article in nonlinear dynamics* ,(**2016**), pp-1-19.

[21].R. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automatic Control*, vol. 20, (**1975**) pp. 1–10.

[22].T. Hinamoto, "2-D Lyapunov equation and filter design based on the Fornasini-Marchesini second model," *IEEE Trans. Circuits Syst. I*, vol. 40, (**1993**), pp. 102–110.

[23].M. Bisiacco, "New results in 2D optimal control theory," *Multidimensional Syst. Signal Process.*, vol. 6, (**1995**), pp. 189–222.

[24]. C. Du and L. Xie, "*H∞ Control and Filtering of Two-Dimensional Systems*", New York, NY, USA: Springer-Verlag, (**2002).**

[25]. H. Kar and V. Singh, "An improved criterion for the asymptotic stability of 2-D digital filters described by the Fornasini-Marchesini second model using saturation arithmetic," *IEEE Trans. Circuits Syst. I*, vol. 46, (**1999**), pp. 1412–1413.

[26]. H. Kar and V. Singh, "Robust stability of 2-D discrete systems described by the Fornasini-Marchesini second model employing quantization/overflow nonlinearities," *IEEE Trans. Circuits Syst. II*, vol. 51, (**2004**), pp. 598–602.

[27].C. K. Ahn, P. Shi, M. V. Basin, "Two-Dimensional Dissipative Control and Filtering for Roesser model, *IEEE Transactions on Automatic Control*, vol. 60, no.7,(**2015**), pp.1745-1759.