Characterization of Sectionally Pseudo Implicative semi lattice.

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ABSTRACT: *The several basic results concerning Congruence kernel of pseudo complemented semi lattice will also exists in sectionally pseudo implicative semi lattice. This manuscript presents necessary and sufficient conditions such that any nonempty subset of sectionally pseudo implicative semi lattice which satisfies these conditions is kernel of some congruence, and also it institutes the notion of perfect kernel in sectionally pseudo implicative semi lattic*

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INTRODUCTION:

An implication semi lattice is an algebraic system which is model logical system equipped with implication (→) and conjunction (∧) but not possessing a disjunction (∨). The implicative semi lattices in algebraic logic is explained by [8]. In the Terminology of [6] an implicative lattice would be called a relatively pseudo complemented lattice.The sectionally pseudo complements give rise to an implication like operation on a sectionally pseudo complemented poset P, which coincides with the relative pseudo complementation if P is distributive. Cornish [10] investigated a congruence relation on pseudo complemented distributive lattices and identified those ideals that are congruence kernels. Blyth [9] showed results concerning congruence kernels and co kernels hold in semi lattice and therefore they do not depend on distributivity nor on the existence of unions. In this manuscript we show that the several basic results concerning Congruence kernel of pseudo complemented semi lattice will also exists in sectionally pseudo implicative semi lattice. Also presents necessary and sufficient conditions such that any nonempty subset of sectionally pseudo implicative semi lattice which satisfies these conditions is kernel of some congruence, and also it institutes the notion of perfect kernel in sectionally pseudo implicative semi lattice

Preliminaries:

1.0 Gratzer[7]: **Definition:** An element a^* is pseudo complement of an element a if and only if $a \wedge a^* = 0$ and $a \wedge x = 0$, implies $x \leq a^*$.

1.1. Pseudo complement semi lattice:- A meet semi lattice S with 0 is said to be pseudo complemented semi lattice if and only if for a in S, a^* is a pseudo complement of a in S.

1.2. Remark: Every pseudo complemented semi lattice is a sectionally pseudo complemented semi lattice.

1.3. Gratzer[7] **Theorem:** Let L be a pseudo complemented semi lattice $S(L)= \{ a^* / a \in L\}$ (be set of all pseudo complements). Then the partial ordering of L partially orders S(L) and makes S(L) in to a Boolean lattice for a, b ∈ S(L), we have $a \wedge b \in S(L)$ and join in S(L) is described by $a \vee b = (a^* \wedge b^*)^*$.

1.4. Gratzer [7]. **Result:** Let L be a pseudo complemented semi lattice and let a, b ∈ L, then $(a\wedge b)^{*} = (a^{**} \wedge b)^{*} = (a^{**} \wedge b^{**})^{*}$.

1.5.Implicative semi lattice: An implicative semi lattice is a system (S, ≤,∧, *,0) in which S is a nonempty set , ≤ is a partial order on S, ∧ is a greatest lower bound with respective ≤ and * is a binary composition in S, such that for any element x, y, z of S, z≤ x * y if and only if z∧ x ≤ y. The operation * is called implicatication.

2.1 Pseudo implicative semi lattice:- An implicative semi lattice S is said to be pseudo implicative semilattice if for any elements x, y, z of implicative semi lattice S if z∧ x ≤ y then z ≤ x * y.

2.2. Theorem:- Every element of pseudo implicative semi lattice is a pseudo complemented element.

Proof:- For any x, y, z in S, where S is a pseudo implicative semi lattice, we have if z∧ x ≤ y., then z≤ x * y . Now if y ∈ S is a pseudo complement element of x in S, then by definition of pseudo complement we have x∧ y =0 and x∧ (z ∧ x) =0, implies z ∧ x ≤ y. Now z ∧ x = x ∧ (z∧ x)= 0 ≤ y. then z ≤ x * y. Therefore every element of S is a pseudo complemented element.

2.3. Sectionally pseudo implicative semi lattice: A meet semi lattice S with 0 is called a sectionally pseudo implicative semi lattice if and only if for every a in S the interval [0,a] is a pseudo implicative semi lattice.

2.4. Theorem: Every pseudo implicative semi lattice is sectionally pseudo implicative semi lattice.

Proof: Suppose S be a pseudo implicative semi lattice, then for any x, y, z in S, if z ∧ x ≤ y then z ≤ x * y . Since every element of pseudo implicative semi lattice is pseudo complemented element, thus x ∧ z = 0 if and only if x≤ y, where y is a pseudo complement of x in S. Let [0 a] ⊆ S. To prove that S is sectionally pseudo implicative semi lattice , we prove that [0 , a] is pseudo implicative semi lattice. Now , let x ∈ [0, a] ⊆ S, then 0 ≤x ≤ a, implies x ∧ a = x, for x, y, z ∈ [0, a] ⊆ S, if z ∧ x ≤ y or z ∧ x ≤ a ∧ x = x and z ∧ x ≤ y , implies z ∧ x ≤ x ∧ y. Therefore [0 a] is a pseudo implicative semi lattice. Hence S is sectionally pseudo implicative semi lattice. Conversely, suppose that semi lattice S = { 0,a,b ,c,d, ≤ , *} be sectionally pseudo implicative semi lattice , in which the intervals [0 ,a]; [0, b] and [0, c] and [0 ,d] are all pseudo implicative semi lattices, but S need not be pseudo implicative semi lattice because an element b may not be pseudo complement of an element a as if a∧ b =0 and a∧ c =0, but c≰ b. Therefore S need not be pseudo implicative semi lattice S.

2.5.Binary relation: Let (S, ≤,∧, *,0) be a sectionally pseudo implicative semi lattice S and R be a binary relation on S denoted by $[0]_R$ and defined by $[0]_R$ = { x in S/ < x, 0> ∈ R}.

2.6.Congruence kernel: If the binary relation R is a congruence on S, then $[0]_R$ is called a congruence kernel of R.

2.7.Example: Let (S, ≤,∧, *,0) be a sectionally pseudo implicative semi lattice S. Now for x, y , z in S, define a relation R as follows x R y if and only if z ∧ x ≤ y. Since z ∧ x ≤ x for all x , we have x R x. Thus R is reflexive relation. Let x R y, then z ∧ x ≤ y since z ∧ x ≤ z , we have (z ∧ x) ∧ y ≤ z ∧ y, implies (z ∧ y) ∧ x ≤ z ∧ y ≤ x, implies z ∧y ≤ x. Thus y R x. Therefore R is symmetric relation. Let x R y and y R z , then z ∧ x ≤ y and x ∧ y ≤ z, thus y ∧ x ≤ z. Therefore x R z. Therefore R is a transitive relation. Hence a relation R is congruence.

2.8. Definition:- A sectionally pseudo implicative semi lattice is said to be almost congruence if x ≡ y, then $x^{*} \equiv y^{*}$ for all x, y in S.

2.9. Theorem: If S is sectionally pseudo implicative semi lattice, then a congruence ≡ on S is a almost congruence if and only if $x \equiv 0$, implies $x^* \equiv 1$.

Proof:- Let S be sectionally pseudo implicative semi lattice and is almost congruence, then if $x \equiv y$, then $x^* \equiv y^*$ for all x, y in S. Let x, y ∈ [0, 1] ⊆ S, if S is almost congruence then if $x \equiv y$, then $x^* \equiv y^*$ for all x, y in S, taking y = 0, we have $x \equiv 0$, then $x^* \equiv 0^*$ in S, implies $x^* \equiv 1$. Thus condition is necessary. Conversely, suppose that condition holds in S, i.e.., if $x \equiv 0$ implies $x^* \equiv 1$; then to prove that S is almost congruence, let $x \equiv y$ for x, y ∈ S. if x* is pseudo complement of x in S then $x \wedge x^* = 0$, thus $0 = x \wedge x^* \equiv y \wedge x^*$; implies $x^* \wedge y \equiv 0$. Therefore $(x^* \wedge y)^* \equiv 1$. Now $1 \equiv x^* = x^* \wedge x^* \equiv x^* \wedge 1 \equiv x^* \wedge (x^* \wedge y)^* = x^* \wedge y^*$. Therefore $x^* \equiv x^* \wedge y^*$. Similarly $y^* \equiv x^* \wedge y^*$. Hence $x^* \equiv y^*$.

2.10. Definition: Perfect set:- A non empty subset I of sectionally pseudo implicative semi lattice S is called a perfect set if (i) for x, y in I, x∧y ∈ I (ii) for x in I there exists t in S such that $x \leq t$ implies t ∈ I.

2.11 Definition: A perfect set I in sectionally pseudo implicative semi lattice S is a perfect kernel if I is a kernel of almost congruence on S.

2.12. Theorem:- A non empty subset I of Sectionally pseudo implicative semi lattice S, is a kernel of some congruence on S if and only if I satisfies the following two conditions(i) If x∈ I and a∈ S, then $x \wedge a$ ∈ I (ii) if x, y in I the$(x^* \wedge y^*)^*$ ∈ I.

Proof:- Let a non empty subset I of Sectionally pseudo implicative semi lattice I of S be a kernel of some congruence on S. If $I = [0]_\theta$ for some θ, be the kernel of some congruence on S, then for x, y in I we have $x \equiv 0$ and $y \equiv 0$ and for a in S, we have $x \wedge a \equiv 0 \wedge a = 0$, implies $x \wedge a \equiv 0$, implies $x \wedge a \in I$, which is condition (i) since $x \equiv 0$ and $y \equiv 0$, we have $x^* \equiv 1$ and $y^* \equiv 1$ in S. Then $x^* \wedge y^* \equiv 1$, which implies $(x^* \wedge y^*)^* \equiv 0$. Thus $(x^* \wedge y^*)^* \in I$ which is condition (ii). Conversely, suppose that I is a non empty subset of S satisfying the conditions (i) and (ii). Now define a relation R on S as follows, x R y if and only if $x \wedge y^* \in I$ which is right compatibility and x R y if and only if $x^* \wedge y \in I$, which is left compatibility since $x^* \wedge x = 0 \in I$ and $x \wedge x^* = 0 \in I$ which implies x R x, so R is reflexive. Now for $x \wedge y^* \in I$ and $x^* \wedge y \in I$, by condition (ii) we have $((x\wedge y^*)^* \wedge (x^* \wedge y)^*)^* \in I$, implies $(x \wedge y^*)^{**} \in I$ and $(x^* \wedge y)^{**} \in I$, implies $x^{**} \wedge y^{***} = x \wedge y^* \in I$ and $x^{***} \wedge y^{**} = x^* \wedge y \in I$, which implies x* R y* and y* R x*. Thus R is symmetric. Now, let x R y and y R z, then $x \wedge y^* \in I$ and $x^* \wedge y \in I$, also $y\wedge z^* \in I$ and $y^* \wedge z \in I$. by condition (ii) we have $((x\wedge y^*)^* \wedge (x^* \wedge y)^*)^* \in I$ and $(y \wedge z^*)^* \wedge (y^* \wedge z)^*) \in I$, implies x* R y* and y* R z*. Now x R z if and only if $x \wedge z^* \in I$ and $x^* \wedge z \in I$, implies $(x \wedge y^*) \wedge (y\wedge z^*) \in I$, as $x\wedge y^*$ and $y\wedge z^* \in I$, implies $x \wedge y^* \wedge y \wedge z^* = x \wedge z^* \in I$. similarly $x^* \wedge y \in I$ and $y^* \wedge z \in I$, implies $x^* \wedge y \wedge y^* \wedge z \in I$, implies $x^* \wedge y \wedge y^* \wedge z = x^* \wedge z \in I$. Therefore x R z. Hence R is transitive relation on S. If x ∈ I, then $x \wedge 0^* = x \wedge 1 = x \in I$ and $x^* \wedge 0 = 0 \in I$, thus x R 0 , i.e.., $x \in [0]_R$, which implies $I \subseteq [0]_R$. Let $x \in [0]_R$, then x R 0, implies $x = x \wedge 0^* \in I$, which implies $[0]_R \subseteq I$. Therefore $I = [0]_R$. Therefore $I =[0]_{\theta(R)}$. Hence I is a kernel of some congruence on S.

2.13. Theorem: A perfect set I of Sectionally pseudo implicative semi lattice S is perfect kernel of S if and only if $(x^* \wedge y^*)^* \in I$.

Proof: Let a perfect Set I of sectionally pseudo implicative semi lattice S is perfect kernel of S, then I is a kernel of almost congruence on S. Thus if x, y ∈ I and $x \equiv 0$ implies $x^* \equiv 1$. Similarly $y \equiv 0$, implies $y^* \equiv 1$. Therefore $x^* \equiv y^* \equiv 1$. Therefore $x^* \wedge y^* \equiv 1$, which implies $(x^* \wedge y^*)^* \in I$. Let us suppose for x, y in S, implies $(x^* \wedge y^*)^* \in I$. To prove that a perfect set I of sectionally pseudo implicative semi lattice S is a perfect kernel of S. Now define a relation ∼ on S by x ∼ y if and only if there exists i in I such that $x \wedge i^* = y \wedge i^*$. Since $x \wedge i^* = x \wedge i^*$ for i in I, then x ∼ x , which represents relation ∼ is reflexive. Let x ∼ y, then $x \wedge i^* = y \wedge i^*$, implies $y \wedge i^* = x \wedge i^*$, implies y ∼ x. Therefore relation ∼ is symmetric. Let x ∼ y and y ∼ z, then $x \wedge i^* = y \wedge i^*$; $y \wedge j^* = z \wedge j^*$ for i, j in I as condition holds for i, j in I ($i^* \wedge j^*)^* \in I$. Let $(i^* \wedge j^*)^* \in I$, now $x \wedge k^* = x \wedge (i^* \wedge j^*)^* = y \wedge (i^* \wedge j^*)^* = z \wedge (i^* \wedge j^*)^*$, which shows $x \wedge k^* = z \wedge k^*$, therefore x ∼ z. Hence relation is transitive. Therefore the relation $\sim$ is an equivalence relation, is a semi lattice congruence on S. To show that I is a kernel of almost congruence on S, since the condition $(x^* \wedge y^*)^* \in I$ holds, by taking $x = y$, we have $(x^* \wedge x^*)^* \in I$, implies $x^{**} \in I$. thus for $x \sim 0$, we have $x \wedge i^* = 0 \wedge i^* = 0$, implies $x \leq i^{**}$ if and only if $x \in I$. i.e.., $x \equiv 0$. Also if $x \sim 1$, then $x \wedge 1^* = 1 \wedge i^* = i^*$ if and only if $x \wedge i^* = i^*$ if and only if $x \geq i^*$ if and only if $x^* \geq i^{**}$ if and only if $i \leq x^*$, implies $x^* \sim 1$, which implies $x^* \equiv 1$. Therefore if S is pseudo implicative semi lattice then a semi lattice congruence on S is almost congruence if and only if $x \equiv 0$ implies $x^* \equiv 1$. Hence the relation $\sim$ is almost congruence on S.

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